PARABOLIC INEQUALITIES IN $L^1$ AS LIMITS OF RENORMALIZED EQUATIONS

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The paper deals with the existence of solutions of some parabolic bilateral problems approximated by the renormalized solutions of some parabolic equations.

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and $T > 0$. We denote by $Q$ the cylinder $\Omega \times (0, T)$ and $\Gamma = \partial Q$.

Let

$$A(u) = -\text{div} (a(x,t,u,\nabla u))$$

be a Leray-Lions operator acting on $L^p(0, T; W^{1,p}_0(\Omega))$, $1 < p < \infty$, into its dual $L^{p'}(0, T; W^{-1,p'}(\Omega))(1/p + 1/p' = 1)$. Consider the following parabolic problem:

$$u \in \mathcal{H} = \{ v \in L^p(0, T; W^{1,p}_0(\Omega)) : v(t) \in K \text{ a.e.} \},$$

$$\int_0^T \langle \frac{\partial v}{\partial t}, u - v \rangle dt + \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla v)dx dt \leq \int_0^T \langle f, u - v \rangle dt,$$

$$\forall v \in \mathcal{H} \cap \{ v \in L^p(0, T; W^{1,p}_0(\Omega)) : \frac{\partial v}{\partial t} \in L^{p'}(0, T; W^{-1,p'}(\Omega)); v(0) = 0 \},$$

where $K$ is a given convex in $W^{1,p}_0(\Omega)$ and $f \in L^{p'}(0, T; W^{-1,p'}(\Omega))$.

It is well known that ($P$) admits at least one solution via a classical penalty method (see Lions [5] for $p \geq 2$ and Landes-Mustonen [4] for $1 < p < 2$). Recently in [6], the authors
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approximated $(P)$ by the following sequence of parabolic equations:

$$
\frac{\partial u_n}{\partial t} + A(u_n) + |h(x,u_n)|^{n-1}h(x,u_n)|G(x,t,u_n,\nabla u_n)| = f \quad \text{in } Q,
$$

$$
u_n(x,t) = 0 \quad \text{on } \partial Q,
$$

$$
u_n(x,0) = 0 \quad \text{in } \Omega,$n

where $h$ and $G$ are two Carathéodory functions satisfying some natural growth conditions. The obtained convex $K$ depends on two obstacles constructed from $h$.

In the $L^1$ case, that is, $f \in L^1(\Omega \times ]0,T[)$, the formulations $(P)$ and $(P_n)$ are not appropriate. So, we introduce the renormalized problem $(R_n)$ associated to $(P_n)$ (see the definition below). The study of the asymptotic behavior of $(R_n)$ as $n \to \infty$ leads to some bilateral parabolic problem. Our approach allows us also to prove the existence of solutions for general parabolic inequalities of type

$$
T_k(u) \in \mathcal{K},
$$

$$
\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle dt + \int_Q a(x,t,u,\nabla u) \nabla T_k(u - v) dx dt
$$

$$
+ \int_Q H(x,t,u,\nabla u) T_k(u - v) dx dt \leq \int_Q f T_k(u - v) dx dt, \quad \forall v \in \mathcal{K} \cap D \cap L^\infty(Q),
$$

(1.2)

where $D = \{v \in L^p(0,T;W^{1,p}_0(\Omega)) \mid \partial v/\partial t \in L^p(0,T,W^{-1,p}_0(\Omega)) + L^1(Q), v(0) = 0\}$ and where $H$ is a given Carathéodory function satisfying some natural growth assumption.

For some recent and classical results for some parabolic inequalities problems, the reader can refer to [2, 7, 9, 10].

2. Main result

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$ and $1 < p < +\infty$.

We denote by $Q$ the cylinder $\Omega \times (0,T)$ and $\Gamma = \partial Q$.

Let $A(u) = -\text{div}(a(x,t,\nabla u))$ be a Leray-Lions operator defined on $L^p(0,T;W^{1,p}_0(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$, where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$ and for all $\zeta, \zeta' \in \mathbb{R}^N$, ($\zeta \neq \zeta'$) the following hold:

$$
|a(x,t,\zeta)| \leq \beta (|x| + |\zeta|^{p-1}),
$$

$$
(a(x,t,\zeta) - a(x,t,\zeta'))(\zeta - \zeta') > 0,
$$

$$
a(x,t,\zeta)\zeta \geq \alpha |\zeta|^p,
$$

(2.1)

with $\alpha > 0$, $\beta > 0$, $k \in L^{p'}(Q)$.

Furthermore, let $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that

$$
h(x,0) = 0, \quad h(x,s) \text{ is nondecreasing with respect to } s.
$$

(2.2)
Theorem 2.1. Thanks to [8, Theorem 3.2, page 164], there exists at least one solution where b is a Carathéodory function satisfying the following assumptions:

\[ |G(x,t,s,ξ)| \leq b(|s|)(c(x,t) + |ξ|^p), \quad G(x,t,s,0) = 0, \quad (2.3) \]

\[ \{v \in L^p(0,T;W^{1,p}_0(\Omega)) : G(x,t,v,\nabla v) = 0 \text{ a.e. in } Q \} \]

\[ \subset \{v \in L^p(0,T;W^{1,p}_0(\Omega)) : |h(x,v)| \leq 1 \text{ a.e. in } Q \}. \quad (2.4) \]

Let us suppose for almost \( x \in \Omega \setminus \Omega^\infty \) there exists \( \epsilon = \epsilon(x) > 0 \) such that

\[ h(x,s) > 1, \quad \forall s \in [q_+(x),q_+(x) + \epsilon[, \quad (2.5) \]

and for almost \( x \in \Omega \setminus \Omega^\infty \) there exists \( \epsilon = \epsilon(x) > 0 \) such that

\[ h(x,s) < -1, \quad \forall s \in [q_-(x) - \epsilon,q_-(x)[, \quad (2.6) \]

where \( b \) is a continuous nondecreasing function and \( c(x,t) \in L^1(Q), c \geq 0, \) and

\[ q_+(x) = \inf \{s > 0, h(x,s) \geq 1\}, \]

\[ q_-(x) = \sup \{s > 0, h(x,s) \leq -1\}, \]

\[ \Omega^\infty_+ = \{x \in \Omega : q_+(x) = +\infty\}, \]

\[ \Omega^\infty_- = \{x \in \Omega : q_-(x) = -\infty\}. \]

We define for all \( s \) and \( k \) in \( \mathbb{R}, k \geq 0, T_k(s) = \max(-k, \min(k,s)) \).

We will say that \( u_n \) is a renormalized solution of \((P_n)\) if

\[ \lim_{h \to 0} \int_{h \leq |u_n| \leq h+1} a(x,t,\nabla u_n) \nabla u_n dx dt = 0, \]

\[ u_n \text{ satisfies in the distributional sense} \]

\[ (A(u_n))_t - \text{div}(a(x,t,\nabla u_n)A'(u_n)) + a(x,t,\nabla u_n) \nabla u_n A''(u_n) \]

\[ + |h(x,u_n)|^{n-1}h(x,u_n) \text{G(x,t,u_n,}\nabla u_n) |A'(u_n)| = fA'(u_n), \quad (R_n) \]

\( R_n \) has a compact support and \( u_n \) satisfies the initial condition in the sense that \( A(u_n) \in C([0,T],L^1(\Omega)) \).

Thanks to [8, Theorem 3.2, page 164], there exists at least one solution \( u_n \) of \( (R_n) \).

**Theorem 2.1.** Under the hypotheses (2.1)–(2.5), \( f \in L^1(Q) \), the problem \((P_n)\) has at least one renormalized solution \((u_n)\) such that

\[ T_k(u_n) \to T_k(u) \quad \text{strongly in } L^p(0,T;W^{1,p}_0(\Omega)), \quad (2.7) \]
where $u$ is a solution of the following obstacle problem:

\[
q_-(x) \leq u(x,t) \leq q_+(x) \quad \text{a.e.} \ (x,t) \in Q,
\]

\[
T_k(u) \in L^p(0,T;W^{1,p}_0(\Omega)),
\]

\[
\int_0^T \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle dt + \int_Q a(x,t,\nabla u) \nabla T_k(u - v) dx \, dt \leq \int_Q f T_k(u - v) dx \, dt, \quad \forall v \in \mathcal{K} \cap D \cap L^\infty(Q),
\]

where

\[
D = \left\{ v \in L^p(0,T;W^{1,p}_0(\Omega)), \frac{\partial v}{\partial t} \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q), v(0) = 0 \right\},
\]

\[
\mathcal{K} = \left\{ v \in L^p(0,T;W^{1,p}_0(\Omega)), v(t) \in K \right\}, \quad K = \left\{ v \in W^{1,p}_0(\Omega), q_- \leq v \leq q_+ \right\}.
\]

Moreover, if $q_-, q_+ \in L^\infty(\Omega)$, then $u \in L^p(0,T;W^{1,p}_0(\Omega)) \cap L^\infty(Q)$.

Remark 2.2. The same result can be obtained when dealing with general operator of Leray-Lions type depending also on $u$, that is, $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$.

Proof of Theorem 2.1.

Step 1. Let $A(t) = H_m(t)$, $H_m(t) = \int_0^t h_m(s) ds$, where

\[
h_m(s) = \begin{cases} 
1 & \text{if } |s| \leq m, \\
\text{affine} & \text{if } m \leq |s| \leq m + 1, \\
0 & \text{if } m + 1 \leq |s|.
\end{cases}
\]

Taking now $T_k(H_m(u_n))$ as test function in (\(R_n\)), we obtain

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n)) \right\rangle dt + \int_{|H_m(u_n)|<k} a(x,t,\nabla u_n) \nabla u_n h_m^2(u_n) dx \, dt \\
+ \int_Q |h(x,u_n)|^{n-1} h(x,u_n) |G(x,t,u_n,\nabla u_n)| h_m(u_n) T_k(H_m(u_n)) dx \, dt \\
+ \int_Q a(\cdot,t,\nabla u_n) \nabla u_n h_m'(u_n) T_k(H_m(u_n)) dx \, dt = \int_Q f h_m(u_n) T_k(H_m(u_n)) dx \, dt.
\]

Since

\[
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n)) \right\rangle dt = \int_{\Omega} \left( \int_0^{H_m(u_n(x,T))} T_k(s) ds \right) dx - \int_{\Omega} \left( \int_0^{H_m(u_n(x,0))} T_k(s) ds \right) dx
\]
and by using the fact that \( \int_{\Omega} (\int_{0}^{H_m(u_n(x,T))} T_k(s)ds) \geq 0 \), we obtain

\[
\int_{\{|H_m(u_n)| < k\}} a(x,t,\nabla u_n) \nabla u_n h^2_m(u_n) \, dx \, dt \leq Ck + \int_{|m_n| \leq m + 1} a(x,t,\nabla u_n) \nabla u_n \, dx \, dt,
\]

\[
\int_{Q} |h(x,u_n)|^{n-1} h(x,u_n) \, |G(x,t,u_n,\nabla u_n)| \, h_m(u_n) T_k(H_m(u_n)) \, dx \, dt \\
\leq Ck + \int_{|m_n| \leq m + 1} a(x,t,\nabla u_n) \nabla u_n \, dx \, dt.
\]  

(2.12)

We have \( H_m(s) \) (resp., \( h_m(s) \)) tends to \( s \) (resp., to 1) as \( m \) goes to \( +\infty \).

Using Fatou’s lemma and the definition of the renormalized solution leads to

\[
\int_{Q} |\nabla T_k(u_n)|^p \, dx \, dt \leq Ck,
\]

(2.13)

\[
\int_{Q} |h(x,u_n)|^{n-1} h(x,u_n) \, |G(x,t,u_n,\nabla u_n)| \, T_k(u_n) \, dx \, dt \leq Ck,
\]

(2.14)

which gives

\[
\int_{Q} |h(x,u_n)|^n \, |G(x,t,u_n,\nabla u_n)| \, \left| \frac{T_k(u_n)}{k} \right| \, dx \, dt \leq C,
\]

(2.15)

and as \( k \to 0 \) we obtain

\[
\int_{Q} |h(x,u_n)|^n \, |G(x,t,u_n,\nabla u_n)| \, dx \, dt \leq C.
\]

(2.16)

Choosing now a \( C^2 \) function \( \rho_k \), such that \( \rho_k(s) = s \) for \( |s| \leq k \) and \( 2k \text{sign}(s) \) for \( |s| > 2k \), we get

\[
\begin{align*}
\left( \rho_k(u_n) \right)_t - \text{div} \left( a(x,t,\nabla u_n) \rho_k'(u_n) \right) + a(x,t,\nabla u_n) \nabla u_n \rho_k''(u_n) \\
+ |h(x,u_n)|^{n-1} h(x,u_n) \, |G(x,t,u_n,\nabla u_n)| \, \rho_k'(u_n) = f \rho_k(u_n).
\end{align*}
\]  

(2.17)

We deduce that \( (\rho_k(u_n))_t \) is bounded in \( L^1(Q) + L^{p'}(0,T; W^{-1,p'}(\Omega)) \).

Now thanks to the following result.

**Lemma 2.3** \([11]\). Let \( p > 1 \). If \( (u_n) \) is a bounded sequence of \( L^p(0,T; W^{1,p}_0(\Omega)) \) such that \( \partial u_n/\partial t \) is bounded in \( L^1 + L^{p'}(0,T; W^{-1,p'}(\Omega)) \), then \( u_n \) is relatively compact in \( L^p(Q) \).
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We deduce that $\rho_k(u_n)$ is relatively compact in $L^p(Q)$ and so there exists a measurable function $u$ such that $u_n \to u$ a.e. in $Q$.

Finally, we deduce from (2.13) that $T_k(u_n) \to T_k(u)$ weakly in $L^p(0,T;W^{1,p}_0(\Omega))$, and strongly in $L^p(Q)$.

Step 2. We are dealing now with the almost convergence of the gradient.

We have to prove that, for $0 < \theta < 1$,

$$\lim_{n \to \infty} \int_Q \left( [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \right)^\theta \, dx \, dt = 0. \quad (2.18)$$

Let $\omega \in L^p(0,T;W^{1,p}_0(\Omega))$, we define for any $\mu > 0$, $\omega_\mu$ the time regularization of $\omega$,

$$\omega_\mu(x,t) = \mu \int_{-\infty}^t \tilde{\omega}(x,s) \exp(\mu(s-t)) \, ds, \quad (2.19)$$

where $\tilde{\omega}$ is the zero extension of $\omega$ for $s > T$. Furthermore, $\omega_\mu$ satisfies the following properties (see [3]):

$$\omega_\mu \rightharpoonup \omega \quad \text{strongly in } L^p(0,T;W^{1,p}_0(\Omega)), \quad (2.20)$$

Letting $\eta > 0$, we obtain

$$\int_Q \left( [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \right)^\theta \, dx \, dt \leq C \, \text{meas} \, \{ |T_k(u_n) - T_k(u)| \geq \eta \}^{1-\theta}$$

$$+ C \left( \int_{|T_k(u_n) - T_k(u)| < \eta} [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] [\nabla T_k(u_n) - \nabla T_k(u)] \right)^\theta. \quad (2.21)$$
On the other hand, we have

\[
\int_{\{T_k(u_n) - T_k(u)_\mu < \eta\}} \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \, dt \\
\leq \int_{\{T_k(u_n) - T_k(u)_\mu < \eta\}} \left[ a(x, t, \nabla T_k(u_n)) \right] - a(x, t, \nabla T_k(u)) \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \, dt \\
+ \int_{\{T_k(u_n) - T_k(u)_\mu < \eta\}} a(x, t, \nabla T_k(u_n)) \left( \nabla T_k(u)_\mu - \nabla T_k(u) \right) dx \, dt \\
+ \int_{\{T_k(u_n) - T_k(u)_\mu < \eta\}} \left[ a(x, t, \nabla T_k(u)_\mu) - a(x, t, \nabla T_k(u)) \right] \nabla T_k(u) dx \, dt \\
- \int_{\{T_k(u_n) - T_k(u)_\mu < \eta\}} a(x, t, \nabla T_k(u)_\mu) \nabla T_k(u)_\mu dx \, dt \\
+ \int_{\{T_k(u_n) - T_k(u)_\mu < \eta\}} a(x, t, \nabla T_k(u)) \nabla T_k(u) dx \, dt \\
\leq I_1 + I_2 + I_3 + I_4 + I_5.
\]

Take \( T_\eta(H_m(u_n) - T_k(u)_\mu) \) as test function in \((R_n)\) with \( A(t) = H_m(t) \). We obtain

\[
\int_0^T \left< \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu) \right> dt \\
+ \int_{\{H_m(u_n) - T_k(u)_\mu < \eta\}} a(x, t, \nabla u_n) \nabla u_n h_m^2(u_n) - \nabla T_k(u)_\mu h_m(u_n) dx \, dt \\
+ \int_Q \left| h(x, u_n) \right|^{n-1} h(x, u_n) \left| G(x, t, u_n, \nabla u_n) \right| h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu) dx \, dt \\
+ \int_Q a(x, t, \nabla u_n) \nabla u_n h_m'(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu) dx \, dt \\
= \int_Q f h_m(u_n) T_\eta(H_m(u_n) - T_k(u)_\mu) dx \, dt.
\]

We have

\[
\int_0^T \left< \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u)_\mu) \right> dt \\
= \int_0^T \left< \frac{\partial H_m(u_n)}{\partial t} - T_k(u)_\mu, T_k(H_m(u_n) - T_k(u)_\mu) \right> dt \\
+ \int_0^T \left< \frac{\partial T_k(u)_\mu}{\partial t}, T_k(H_m(u_n) - T_k(u)_\mu) \right> dt.
\]
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Using the fact that

$$\int_0^T \left< \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u_\mu)) \right> dt \geq 0,$$

$$\int_0^T \left< \frac{\partial T_k(u_\mu)}{\partial t}, T_k(H_m(u_n) - T_k(u_\mu)) \right> dt = \mu \int_Q (T_k(u) - T_k(u_\mu)) T_\eta(u - T_k(u_\mu)) dx dt,$$

consequently,

$$\limsup_{n \to \infty} \limsup_{m \to \infty} \int_0^T \left< \frac{\partial H_m(u_n)}{\partial t}, T_\eta(H_m(u_n) - T_k(u_\mu)) \right> dt \geq \mu \int_Q (T_k(u) - T_k(u_\mu)) T_\eta(u - T_k(u_\mu)) dx dt = \epsilon(m,n) \geq 0.$$  \hfill (2.26)

This implies that

$$\int_{\{|H_m(u_n) - T_k(u_\mu)| < \eta\}} a(x,t,\nabla u_n) \nabla u_n h_m^2(u_n) - \nabla T_k(u_\mu) h_m(u_n) dx dt$$

$$+ \int_Q \left| h(x,u_n) \right| \left| h(x,u_n) \right|^2 G(x,t,u_n,\nabla u_n) \left| h_m(u_n) T_\eta(H_m(u_n) - T_k(u_\mu)) dx dt \right.$$  \hfill (2.27)

$$+ \int_Q a(x,t,\nabla u_n) \nabla u_n h_m'(u_n) T_\eta(H_m(u_n) - T_k(u_\mu)) dx dt$$

$$\leq \int_Q f h_m(u_n) T_\eta(H_m(u_n) - T_k(u_\mu)) dx dt + \epsilon(m,n),$$

which gives by using the fact that

$$\int_Q \left| h(x,u_n) \right| \left| h(x,u_n) \right|^2 G(x,t,u_n,\nabla u_n) \left| h_m(u_n) T_\eta(H_m(u_n) - T_k(u_\mu)) dx dt \right. \leq C \eta,$$  \hfill (2.28)

$$\int_{\{|H_m(u_n) - T_k(u_\mu)| < \eta\}} a(x,t,\nabla u_n) \nabla u_n h_m^2(u_n) - \nabla T_k(u_\mu) h_m(u_n) dx dt$$

$$\leq C \eta + \epsilon(m,n) + \eta \int_{\{|m\leq|u_n|\leq m+1\}} a(x,t,\nabla u_n) \nabla u_n dx dt,$$

which gives as $m \to \infty$,

$$\int_{\{|u_n - T_k(u_\mu)| < \eta\}} a(x,t,\nabla u_n) \nabla u_n - \nabla T_k(u_\mu) dx dt \leq C \eta + \epsilon(n).$$  \hfill (2.29)
Finally from (2.22),
\[ |I_1| \leq C\eta + \varepsilon(n) - \int_{|T_k(u_n) - T_k(u)| < \eta} a(x, t, \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)). \] (2.30)

Since \( a(x, t, \nabla T_k(u)) \chi_{|T_k(u_n) - T_k(u)| < \eta} \rightarrow a(x, t, \nabla T_k(u)) \chi_{|T_k(u) - T_k(u)| < \eta} \) in \( L^p(Q) \) and \( T_k(u_n) \rightarrow T_k(u) \) weakly in \( L^p(0, T; W^{1,p}_0(\Omega)) \), then
\[ -\int_{|T_k(u_n) - T_k(u)| < \eta} a(x, t, \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) \, dx \, dt \]
\[ = -\int_{|T_k(u) - T_k(u)| < \eta} a(x, t, \nabla T_k(u)) (\nabla T_k(u) - \nabla T_k(u)) \, dx \, dt + \varepsilon(n). \] (2.31)

So
\[ |I_1| \leq C\eta + \varepsilon(n). \] (2.32)

For what concerns the term \( I_2 \), one has
\[ I_2 = \varepsilon(n, \mu), \] (2.33)

since
\[ a(x, t, \nabla T_k(u_n)) \chi_{|T_k(u_n) - T_k(u)| < \eta} \rightarrow a(x, t, \nabla T_k(u)) \chi_{|T_k(u) - T_k(u)| < \eta} \] in \( (L^p(Q))^N \),
\[ (\nabla T_k(u_n) - \nabla T_k(u)) \chi_{|T_k(u_n) - T_k(u)| < \eta} \rightarrow (\nabla T_k(u_n) - \nabla T_k(u)) \chi_{|T_k(u) - T_k(u)| < \eta}. \] (2.34)

In the same way, we show that
\[ I_3 = \varepsilon(n, \mu), \quad I_4 = \varepsilon(n, \mu), \quad I_5 = \varepsilon(n, \mu). \] (2.35)

Combining the above estimates, we get
\[ \lim_{n \to \infty} \int_Q \left[ a(x, t, \nabla T_k(u_n)) - a(x, t, \nabla T_k(u)) \right] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right]^\theta \, dx \, dt = 0. \] (2.36)

Then there exists a subsequence also denoted by \( (u_n) \) such that
\[ \nabla u_n \rightharpoonup \nabla u \quad \text{a.e. in } Q. \] (2.37)

**Step 3.** From (2.16), we deduce that
\[ \int_Q |h(x, u_n)\n\|G(x, t, u_n, \nabla u_n)| \, dx \, dt \leq C, \] (2.38)

which gives for every \( \beta > 0 \),
\[ \int_{|h(x, T\beta(u_n))| > k} |G(x, t, T\beta(u_n), \nabla T\beta(u_n))| \, dx \, dt \leq \frac{C}{k^\beta}, \] (2.39)
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where $k > 1$. Letting $n \to +\infty$ for $k$ fixed, we deduce by using Fatou’s lemma

$$\int_{|h(x, T\beta(u))| > k} |G(x, t, T\beta(u), \nabla T\beta(u))| \, dx \, dt = 0, \tag{2.40}$$

and so, by (2.4)

$$|h(x, T\beta(u))| \leq 1 \quad \text{a.e. in } Q. \tag{2.41}$$

So

$$q_-(x) \leq T\beta(u(x)) \leq q_+ \quad \text{a.e. in } Q. \tag{2.42}$$

Letting now $\beta \to +\infty$, we deduce also that

$$q_-(x) \leq u(x) \leq q_+ \quad \text{a.e. in } Q. \tag{2.43}$$

**Step 4. Strong convergence of the truncations.**

We will prove that

$$\lim_{n \to \infty} \int_Q \left[ a(x, t, \nabla T_k(u_n)) \right] \frac{\partial H_m(u_n)}{\partial t} - a(x, t, \nabla T_k(u)) \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \, dx \, dt = 0. \tag{2.44}$$

Fix $k > 0$ and let $\varphi(s) = \exp(\delta s^2)$, $\delta > 0$. Let $l > k$ and define the function $R_l(s) = \int_0^s \rho(t) \, dt$. Let us consider $\omega^m = T_k(H_m(u_n))$, where $v_\mu$ is the mollification with respect to time $v$. Letting $v^{m,n}_\mu = \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m)$ as test function in the problem $(R_n)$, we get

$$\int_0^T \left( \frac{\partial H_m(u_n)}{\partial t} - a(x, t, \nabla T_k(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m) \right) \, dt$$

$$+ \int_Q a(x, t, \nabla u_n) \nabla u_n h^2(u_n) \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m) \, dx \, dt$$

$$+ \int_Q a(x, t, \nabla u_n) \left( \nabla T_k(H_m(u_n)) - \nabla \omega^m \right)$$

$$\times h_m(u_n) \rho_l(H_m(u_n)) \varphi'(T_k(H_m(u_n)) - \omega^m) \, dx \, dt$$

$$+ \int_Q a(x, t, \nabla u_n) h_m(u_n) \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m) \, dx \, dt$$

$$+ \int_Q |h(x, u_n)|^{n-1} h(x, u_n) |G(x, t, u_n, \nabla u_n)|$$

$$\times h_m(u_n) \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m) \, dx \, dt$$

$$= \int_Q f v^{m,n}_\mu h_m(u_n) \, dx \, dt. \tag{2.45}$$

We deal now with the estimate of each term of the last equalities.
Since $H_m(u_n) \in L^p(0, T; W^{1,p}_0(\Omega))$ and $\partial H_m(u_n)/\partial t \in L^p(0, T; W^{-1, p'}(\Omega)) + L^1(Q)$, there exists a smooth function $H_m(u_n)_\sigma$ such that as $\sigma \to 0$,

$$H_m(u_n)_\sigma \to H_m(u_n) \quad \text{strongly in } L^p(0, T; W^{1,p}_0(\Omega)),$$

$$\frac{\partial H_m(u_n)_\sigma}{\partial t} \to \frac{\partial H_m(u_n)}{\partial t} \quad \text{strongly in } L^p(0, T; W^{-1, p'}(\Omega)) + L^1(Q).$$

This implies that

$$I = \int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_\mu) \right\rangle dt$$

$$= \lim_{\sigma \to 0} \int_Q \left( H_m(u_n)_\sigma \right)' \rho_l(H_m(u_n)_\sigma) \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu) dx \, dt$$

$$= \lim_{\sigma \to 0} \int_Q \left[ R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \right]' \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu) dx \, dt$$

$$+ \int_Q \left[ T_k(H_m(u_n)_\sigma) \right]' \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu) dx \, dt$$

$$= \lim_{\sigma \to 0} \left\{ \int_\Omega \left[ R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu) \right]_0^T dx \, dt$$

$$- \int_Q \left[ R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \right]' \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu)(T_k(H_m(u_n)_\sigma)$$

$$- \omega^m_\mu)' dx \, dt + \int_Q \left[ T_k(H_m(u_n)_\sigma) \right]' \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu) dx \, dt \right\}$$

$$= \lim_{\sigma \to 0} \{ I_1(\sigma) + I_2(\sigma) + I_3(\sigma) \}.$$  (2.47)

Observe that for $|s| \leq k$ we have $R_l(s) = T_k(s) = s$ and for $|s| > k$ we have $|R_l(s)| \geq |T_k(s)|$ and, since both $R_l(s)$ and $T_k(s)$ have the same sign of $s$, we deduce that $\text{sign}(s)(R_l(s) - T_k(s)) \geq 0$. Consequently,

$$I_1(\sigma) = \int_{|H_m(u_n)_\sigma| > k} \left[ R_l(H_m(u_n)_\sigma) - T_k(H_m(u_n)_\sigma) \varphi(T_k(H_m(u_n)_\sigma) - \omega^m_\mu) \right]_0^T dx \, dt \geq 0.$$  (2.48)
We have, since $ (R_l(s) - T_k(s))(T_k(s))' = 0, $ for all $ s $,

$$
I_2(\sigma) = \int_{\{|H_m(u_n)_{\sigma}| > k\}} \left[ R_l(H_m(u_n)_{\sigma}) - T_k(H_m(u_n)_{\sigma}) \right] \varphi'(T_k(H_m(u_n)_{\sigma}))

- \omega^m_{\mu}) (\omega^m_{\mu})' \, dx \, dt

= \mu \int_{\{|H_m(u_n)_{\sigma}| > k\}} \left[ R_l(H_m(u_n)_{\sigma}) - T_k(H_m(u_n)_{\sigma}) \right] \varphi'(T_k(H_m(u_n)_{\sigma}))

- \omega^m_{\mu}) (T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt,

$$

(2.49)

by using the fact that $ \varphi' \geq 0 $ and that

$$
(R_l(H_m(u_n)_{\sigma}) - T_k(H_m(u_n)_{\sigma})) (T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \chi_{\{|H_m(u_n)_{\sigma}| > k\}} \geq 0,

(2.50)

the last integral is of the form $ \epsilon(m,n). $ We deduce that

$$
\lim_{\sigma \to 0^+} \sup I_2(\sigma) \geq \epsilon(m,n).

(2.51)

For $ I_3(\sigma), $ one has

$$
I_3(\sigma) = \int_Q \left[ T_k(H_m(u_n)_{\sigma}) \right]' \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt

= \int_Q \left[ T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu} \right]' \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt

+ \int_Q (\omega^m_{\mu})' \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \, dx \, dt.

(2.52)

Let $ \Phi(s) = \int_0^s \varphi(t) \, dt. $ Remark that $ (T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \varphi(T_k(H_m(u_n)_{\sigma}) - \omega^m_{\mu}) \geq 0. $ Integrating by parts, using the fact that $ \Phi \geq 0, $ and following the same way as above, we have

$$
\lim_{\sigma \to 0^+} \sup I_3(\sigma) \geq \epsilon(m,n).

(2.53)

Combining these estimates, we conclude that

$$
\int_0^T \left\langle \frac{\partial H_m(u_n)}{\partial t}, \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_{\mu}) \right\rangle \, dt \geq \epsilon(m,n).

(2.54)
We set
\[
I_4(m) = \int_Q \left| h(x,u_n) \right| n-1 \left| h(x,u_n) \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega_m^\mu) \right| G(x,t,u_n,\nabla u_n) \right| dx dt,
\] (2.55)
so we have
\[
\limsup_{m \to \infty} I_4(m) \geq I_4^1 + I_4^2,
\] (2.56)
where
\[
I_4^1 = \int_{\{|u_n| < k, 0 \leq u_n \leq T_k(u_\mu)\}} \left| h(x,u_n) \right| n-1 \left| h(x,u_n) \varphi(T_k(u_n) - T_k(u_\mu)) \rho_l(u_n) \right| G(x,t,u_n,\nabla u_n) \right| dx dt,
\]
\[
I_4^2 = \int_{\{|u_n| < k, T_k(u_\mu) \leq u_n \leq 0\}} \left| h(x,u_n) \right| n-1 \left| h(x,u_n) \varphi(T_k(u_n) - T_k(u_\mu)) \rho_l(u_n) \right| G(x,t,u_n,\nabla u_n) \right| dx dt.
\] (2.57)
Since \( q_- \leq T_k(u_\mu) \leq q_+ \) (recall that \( q_- \leq T_k(u) \leq q_+ \)) and \( 0 \leq \rho_l(u_n) \leq 1 \), one easily has
\[
|I_4^1| \leq \int_{\{|u_n| < k\}} c(x,t) |\varphi(T_k(u_n) - T_k(u_\mu)|
\]
\[
+ \frac{b(k)}{\alpha} \int_{\{|u_n| < k\}} \left| \nabla u_n \right|^p |\varphi(T_k(u_n) - T_k(u_\mu)|
\] (2.58)
\[
\leq \frac{b(k)}{\alpha} \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u_\mu))] [\nabla T_k(u_n) - \nabla T_k(u_\mu)]
\]
\[
\times |\varphi(T_k(u_n) - T_k(u_\mu)| dx dt + \epsilon(n,\mu),
\]
and also we have the same estimation of \( I_4^2 \).
Then
\[
|I_4^1| + |I_4^2| \leq 2 \frac{b(k)}{\alpha} \int_Q [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u_\mu))] [\nabla T_k(u_n) - \nabla T_k(u_\mu)]
\]
\[
\times |\varphi(T_k(u_n) - T_k(u_\mu)| dx dt + \epsilon(n,\mu).
\] (2.59)
By denoting by $J_1$ the third term of (2.45), one can write

$$J_1 = \int_Q a(x,t,\nabla u_n) \left( \nabla T_k (H_m(u_n)) - \nabla T_k (H_m(u)) \right) \cdot h_m(u_n) \rho_l(H_m(u_n)) \phi'\left( T_k (H_m(u_n)) - T_k (H(u)) \right) dt$$

$$+ \int_{|u_n| > k} a(x,t,\nabla u_n) \left( \nabla T_k (H_m(u_n)) - \nabla T_k (H_m(u)) \right) h_m(u_n) \rho_l(H_m(u_n)) \phi'\left( T_k (H_m(u_n)) - T_k (H(u)) \right) dt$$

$$= \int_Q a(x,t,\nabla T_k (u_n)) \left( \nabla T_k (H_m(u_n)) - \nabla T_k (H_m(u)) \right) h_m(u_n) \rho_l(H_m(u_n)) \phi'\left( T_k (H_m(u_n)) - T_k (H(u)) \right) dt$$

$$+ \int_{|u_n| > k} a(x,t,\nabla u_n) \left( \nabla T_k (H_m(u_n)) - \nabla T_k (H_m(u)) \right) h_m(u_n) \rho_l(H_m(u_n)) \phi'\left( T_k (H_m(u_n)) - T_k (H(u)) \right) dt.$$

Since $a(x,t,\nabla u_n)\rho_l(u_n)$ is bounded in $L^p(Q)$, we deduce that

$$a(x,t,\nabla u_n)\rho_l(u_n) \rightharpoonup a(x,t,\nabla u)\rho_l(u) \quad \text{weakly in } L^p(Q),$$

and so

$$J_1 = \int_Q \left( a(x,t,\nabla T_k (u_n)) - a(x,t,\nabla T_k (u)) \right) \left( \nabla T_k (u_n) - \nabla T_k (u) \right) \phi'\left( T_k (u_n) - T_k (u) \right) dt + \epsilon(m,n,\mu).$$

Concerning the second term of (2.45), one easily has

$$\int_Q a(x,t,\nabla u_n) \nabla u_n h_m^2(u_n) \rho_l(H_m(u_n)) \phi(T_k (H_m(u_n)) - \omega_m) dt$$

$$\leq \phi(2k) \int_{|u_n| < l+1} a(x,t,\nabla u_n) \nabla u_n dx dt,$$

and since

$$\int_{|u_n| < l+1} a(x,t,\nabla u_n) \nabla u_n dx dt \leq \int_{|u_n| > l} |f| dx dt,$$
we deduce that

\[
\int_{Q} \left| a(x,t,\nabla u_n) \nabla u_n h^2_m(u_n) \rho_l(H_m(u_n)) \varphi(T_k(H_m(u_n)) - \omega^m_k) \right| dx \; dt \\
\leq \varphi(2k) \int_{|u_n| > \ell} |f| dx \; dt = \epsilon(n,l).
\]  

(2.65)

The same result can be obtained for the fourth term of (2.45).

Combining (2.45)–(2.65), using the fact that \( \phi' - 2(b(k) / \alpha) |\phi| \geq 1/2 \) for \( \delta \geq (b(k)/\alpha)^2 \), we deduce that

\[
\lim_{n \to \infty} \int_{Q} [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \; dt = 0. 
\]  

(2.66)

On the other hand, we have

\[
\begin{align*}
&\int_{Q} [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \; dt \\
&\quad - \int_{Q} [a(x,t,\nabla T_k(u_n)) - a(x,t,\nabla T_k(u))] \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \; dt \\
&\quad - \int_{Q} a(x,t,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u_n)) dx \; dt \\
&\quad - \int_{Q} a(x,t,\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u_n)) dx \; dt \\
&\quad + \int_{Q} a(x,t,\nabla T_k(u)) \nabla T_k(u_n)) \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] dx \; dt = \epsilon(n,\mu).
\end{align*}
\]  

(2.67)

Consequently by [1, Lemma 5], we obtain

\[
T_k(u_n) \rightharpoonup T_k(u) \quad \text{strongly in } L^p(0,T; W^{1,p}_0(\Omega)) \quad \text{for every } k > 0.
\]  

(2.68)

Step 5 (passage to the limit). Letting \( v \in D \cap \mathcal{H} \cap L^\infty(Q) \), and using \( T_k(H_m(u_n) - \theta v) \) as test function in the problem \( (R_n) \), we obtain

\[
\begin{align*}
\int_{0}^{T} \left\langle \frac{\partial H_m(u_n)}{\partial t}, T_k(H_m(u_n) - \theta v) \right\rangle dt &+ \int_{Q} a(x,t,\nabla u_n) \nabla T_k(H_m(u_n) - \theta v) h_m(u_n) dx \; dt \\
&+ \int_{Q} a(x,t,\nabla u_n) \nabla u_n T_k(H_m(u_n) - \theta v) h'_m(u_n) dx \; dt \\
&+ \int_{Q} |h(x,u_n)|^{n-1} h(x,u_n) G(x,t,u_n,\nabla u_n) |h_m(u_n) T_k(H_m(u_n) - \theta v) dx \; dt \\
&\leq \int_{Q} f(T_k(H_m(u_n) - \theta v) h_m(u_n) dx \; dt.
\end{align*}
\]  

(2.69)
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We have

$$
\int_Q \left| h(x, u_n) \right|^\alpha h(x, u_n) \left| G(x, t, u_n, \nabla u_n) \right| h_m(u_n) T_k (H_m(u_n) - \theta v) \, dx \, dt \\
\geq \int_{\{0 \leq H_m(u_n) \leq \theta v\}} \left| h(x, u_n) \right|^\alpha \\
\times h(x, u_n) \left| G(x, t, u_n, \nabla u_n) \right| h_m(u_n) T_k (H_m(u_n) - \theta v) \, dx \, dt \\
+ \int_{\{\theta v \leq H_m(u_n) \leq 0\}} \left| h(x, u_n) \right|^\alpha \\
\times h(x, u_n) \left| G(x, t, u_n, \nabla u_n) \right| h_m(u_n) T_k (H_m(u_n) - \theta v) \, dx \, dt.
$$

Now we deal with the estimation of the last two terms in the right-hand side of the last inequality which we denote, respectively, by $J'_1(m, n)$ and $J'_2(m, n)$. Let us define

$$
\delta_1(x, t) = \sup_{0 \leq s \leq \theta v} h(x, s),
$$

then we get $0 \leq \delta_1(x, t) < 1$ a.e. in $Q$.

We have

$$
\limsup_{m \to \infty} \left| J'_1(m, n) \right| \leq k \int_{\{0 \leq u_n \leq \theta v\}} (\delta(x, t))^\alpha (c(x, t) + \left| \nabla u_n \right|^p) \\
\leq \int_{\{|u_n| \leq \|u\|_\infty\}} (\delta(x, t))^\alpha (c(x, t) + \left| \nabla u_n \right|^p),
$$

and by using the strong convergence of $T_{H_m(u_n)}$ in $L^p(0, T; W^{1,p}_0(\Omega))$, we deduce that

$$
\limsup_{n \to \infty} \limsup_{m \to \infty} \left| J'_1(m, n) \right| = 0,
$$

with the same technique (taking $\delta_2(x, t) = \inf_{\theta v \leq s \leq 0} h(x, s)$), we can see that

$$
\limsup_{m \to \infty} \left| J'_2(n, m) \right| \to 0 \quad \text{as } n \to +\infty.
$$

On the other hand,

$$
\int_Q a(x, t, \nabla u_n) \nabla T_k (H_m(u_n) - \theta v) h_m(u_n) \, dx \, dt \\
= \int_Q a(x, t, \nabla u_n) \nabla (H_m(u_n) - \theta v) \chi_{\{|H_m(u_n) - \theta v| \leq k\}} h_m(u_n) \, dx \, dt \\
= \int_Q (a(x, t, \nabla u_n) - a(x, t, \theta \nabla v)) \nabla (H_m(u_n) - \theta v) \chi_{\{|H_m(u_n) - \theta v| \leq k\}} h_m(u_n) \, dx \, dt \\
+ \int_Q a(x, t, \theta \nabla v) \nabla (H_m(u_n) - \theta v) \chi_{\{|H_m(u_n) - \theta v| \leq k\}} h_m(u_n) \, dx \, dt.
$$

(2.75)
Since \( a(x, t, \theta v) \) belongs to \((L^p(Q))^{N} \), using Fatou’s lemma in the first term of the last side gives

\[
\liminf_{n, m \to +\infty} \int_{0}^{T} \langle Au_n, T_k(H_m(u_n) - \theta v) \rangle \, dt \geq \int_{0}^{T} \langle Au, T_k(u) - \theta v \rangle \, dt. \tag{2.76}
\]

Go back to (2.69) and pass to the limit as \( m, n \to \infty \) to obtain

\[
\int_{0}^{T} \left\langle \frac{\partial v}{\partial t}, T_k(u - \theta \nabla v) \right\rangle \, dt + \int_{Q} a(x, t, \nabla u) \nabla T_k(u - \theta v) \, dx \, dt \leq \int_{Q} f T_k(u - \theta v) \, dx \, dt. \tag{2.77}
\]

Letting now \( \theta \) tend to 1, we get

\[
\int_{0}^{T} \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle \, dt + \int_{Q} a(x, t, \nabla u) \nabla T_k(u - v) \, dx \, dt \leq \int_{Q} f T_k(u - v) \, dx \, dt, \tag{2.78}
\]

which completes the proof. \( \square \)

**Remark 2.4.** The same technique allows us to prove an existence result for solutions of the following parabolic inequalities:

\[
q_-(x) \leq u(x, t) \leq q_+(x) \quad \text{a.e. in } Q,
\]

\[
T_k(u) \in L^p(0, T; W_{0}^{1,p}(\Omega)),
\]

\[
\int_{0}^{T} \left\langle \frac{\partial v}{\partial t}, T_k(u - v) \right\rangle \, dt + \int_{Q} a(x, t, \nabla u) \nabla T_k(u - v) \, dx \, dt + \int_{Q} H(x, t, u, \nabla u) T_k(u - v) \, dx \, dt \leq \int_{Q} f T_k(u - v) \, dx \, dt, \quad \forall \, v \in \mathcal{H} \cap D \cap L^{\infty}(Q), \tag{2.79}
\]

where \( H \) is a given Carathéodory function satisfying, for all \((s, \zeta) \in \mathbb{R} \times \mathbb{R}^{N}\) and a.e. \((x, t) \in Q\), the following conditions:

\[
| H(x, t, s, \zeta) | \leq \lambda(|s|) (\delta(x, t) + |\zeta|^{p}),
\]

\[
H(x, t, s, \zeta)s \geq 0, \tag{2.80}
\]

with \( \lambda: \mathbb{R}^{+} \to \mathbb{R}^{+} \) is a continuous increasing function and \( \delta(x, t) \) is a given positive function in \( L^{1}(Q) \).

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Time-Dependent Billiards

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

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