Schur functions and characters of Lie algebras and superalgebras

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Inspired by a conjecture by Joris Van der Jeugt (University of Gent, Belgium) including ongoing joint work with Angèle Hamel (Wilfred Laurier University, Canada)

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Motivation

- Received query from Joris Van der Jeugt (working with Stijn Lievens and Neli Stoilova)
- Studying representations of the orthosymplectic Lie superalgebra $osp(1|2n)$ built using parabosons
- Identified Fock space modules $\overline{V}(p)$ for any $p \in \mathbb{N}$
- Constructed unitary irreducible infinite-dimensional representations $V(p) = \overline{V}(p)/M(p)$ where $M(p)$ is the maximal submodule of $\overline{V}(p)$, and found that
  - for $p \geq n$ irrep $V(p) = \overline{V}(p)$
  - for $p < n$ irrep $V(p) = \overline{V}(p)/M(p)$
- Also calculated the characters of both $\overline{V}(p)$ and $V(p)$
Van der Jeugt’s conjecture

**Proposition** [Van der Jeugt, Lievens and Stoilova, 2007]

Let \( x = (x_1, x_2, \ldots, x_n) \), then

\[
\text{ch } V(p) = (x_1 x_2 \cdots x_n)^{p/2} \sum_{\lambda: \ell(\lambda) \leq p} s_{\lambda}(x)
\]
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\]

**Conjecture** [Van der Jeugt, Lievens and Stoilova, 2007]

\[
\sum_{\lambda: \ell(\lambda) \leq p} s_{\lambda}(x) = \frac{\sum_{\eta} (-1)^{c_\eta} s_{\eta}(x)}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_i x_j)}
\]

with the sum over all partitions \( \eta \) which in Frobenius notation take the form

\[
\eta = \left( \begin{array}{cccc}
    a_1 & a_2 & \cdots & a_r \\
    a_1 + p & a_2 + p & \cdots & a_r + p
\end{array} \right)
\]

with \( c_\eta = (|\eta| - rp + r)/2 \)
Macdonald’s Theorem

- Joris Van der Jeugt asked if the result was known
- If so where could it be found, if not could I supply a proof?
- Angèle Hamel reminded me of:

**Theorem** [Macdonald 79]

\[
\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) = \frac{|x_i^{n-j} - x_i^{n+p+j-1}|}{\prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (x_j - x_k)(1 - x_j x_k)}
\]
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\]

Need to compare this with an immediate **Corollary to Van der Jeugt’s Conjecture**

\[
\sum_{\lambda : \ell(\lambda') \leq p} s_\lambda(x) = \frac{\sum_{\eta} (-1)^{c_\eta} s_{\eta'}(x)}{\prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(1 - x_i x_j)}
\]
Strategy

- Try to recast the numerator of Macdonald’s formula as a signed sum of Schur functions
- Use conjugacy to recover Van der Jeugt’s formula
- Try to identify the origin of the row length restriction \( \ell(\lambda') \leq p \) in Macdonald’s formula
- Try to identify the origin of the column length restriction \( \ell(\lambda) \leq p \) in Van der Jeugt’s Conjecture
Strategy

- Try to recast the numerator of Macdonald’s formula as a signed sum of Schur functions
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- Try to identify the origin of the column length restriction $\ell(\lambda) \leq p$ in Van der Jeugt’s Conjecture
- First some preliminaries on
  - Schur functions and Schur functions series
  - Partitions, Young diagrams, Frobenius notation
  - Determinantal identities and modifications
Schur functions

- Let \( n \) be a fixed positive integer.
- Let \( x = (x_1, x_2, \ldots, x_n) \) be a sequence of indeterminates.
- Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \) be a partition of weight \( |\lambda| \) and length \( \ell(\lambda) \leq n \).
- Then the Schur function \( s_\lambda(x) \) is defined by:

\[
s_\lambda(x) = \left| \frac{x_i^{\lambda_j+n-j}}{x_i^{n-j}} \right|_{1 \leq i, j \leq n} \]

- where \( \left| x_i^{n-j} \right|_{1 \leq i, j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j) \)
Schur functions

Let $n$ be a fixed positive integer

Let $x = (x_1, x_2, \ldots, x_n)$ be a sequence of indeterminates

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

be a partition of weight $|\lambda|$ and length $\ell(\lambda) \leq n$

Then the Schur function $s_\lambda(x)$ is defined by:

$$s_\lambda(x) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

where $|x_i^{n-j}|_{1 \leq i, j \leq n} = \prod_{1 \leq i < j \leq n} (x_i - x_j)$

These Schur functions form a $\mathbb{Z}$-basis of $\Lambda_n$, the ring of polynomial symmetric functions of $x_1, \ldots, x_n$. 
Partitions and Young diagrams

- **Young diagrams** $F^\lambda$ consists of $|\lambda|$ boxes arranged in $\ell(\lambda)$ rows of lengths $\lambda_i$ for $i = 1, 2, \ldots \ell(\lambda)$

- **Conjugate partition** $\lambda'$ is the partition defined by the $\ell(\lambda')$ columns of $F^\lambda$ of lengths $\lambda'_j$ for $j = 1, 2 \ldots, \ell(\lambda')$

- **Frobenius notation** If $F^\lambda$ has $r$ boxes on the main diagonal, with arm and leg lengths $a_k$ and $b_k$ for $k = 1, 2, \ldots, r$, then $\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$ has rank $r(\lambda) = r$ with $a_1 > a_2 > \cdots > a_r \geq 0$ and $b_1 > b_2 > \cdots > b_r \geq 0$
Partitions and Young diagrams

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Special families of partitions

- Let $\mathcal{P}$ be the set of all partitions, including the zero partition $\lambda = 0 = (0, 0, \ldots, 0)$.

- The zero partition is the unique partition of weight, length and rank zero, i.e. $|0| = \ell(0) = r(0) = 0$

- Then for any integer $t$ let

$$
\mathcal{P}_t = \left\{ \lambda = \left( \begin{array}{cc} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{array} \right) \in \mathcal{P} \mid a_k - b_k = t \quad \text{for} \quad k = 1, 2, \ldots, r \right\}
$$

and $r = 0, 1, \ldots$

- Note: The zero partition belongs to $\mathcal{P}_t$ for all integer $t$
Modification rules

For $n \in \mathbb{N}$ let $x = (x_1, x_2, \ldots, x_n)$ and $x = x_1 x_2 \cdots x_n$

Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ with $\kappa_i \in \mathbb{Z}$ for $i = 1, 2, \ldots, n$

Let $s_{\kappa}(x) = \frac{\left| \sum_{1 \leq i,j \leq n} x_i^{\kappa_j+n-j} \right|}{\left| \sum_{1 \leq i,j \leq n} x_i^{n-j} \right|}$

Either $s_{\kappa}(x) = 0$ or $s_{\kappa}(x) = \pm x^k s_{\lambda}(x)$ for some partition $\lambda$ and some integer $k$
Modification rules

For \( n \in \mathbb{N} \) let \( x = (x_1, x_2, \ldots, x_n) \) and \( x = x_1 x_2 \cdots x_n \)

Let \( \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \) with \( \kappa_i \in \mathbb{Z} \) for \( i = 1, 2, \ldots, n \)

Let \( s_\kappa(x) = \sum_{1 \leq i, j \leq n} |x_i^{\kappa_j+n-j}| \) \( x_i^{n-j} \) \( 1 \leq i, j \leq n \)

Either \( s_\kappa(x) = 0 \) or \( s_\kappa(x) = \pm x^k s_\lambda(x) \) for some partition \( \lambda \) and some integer \( k \)

Permuting columns leads to various identities, such as

\[ s_\kappa(x) = -s_\mu(x) \quad \text{and} \quad s_\kappa(x) = (-1)^{j-1} s_\nu(x) \]

with

\[ \mu = (\kappa_1, \ldots, \kappa_{j+1} - 1, \kappa_{j} + 1, \ldots, \kappa_n) \]

and

\[ \nu = (\kappa_{j+1} - j, \kappa_1 + 1, \ldots, \kappa_{j} + 1, \kappa_{j+2} \ldots, \kappa_n) \]
Example

If \( n = 4 \) and \( \kappa = (0, 4, 0, 9) \) then \( s_\kappa(x) = (-1)^{3+1} s_\lambda(x) \)
with \( \lambda = (6, 4, 2, 1) \) since

\[
\begin{vmatrix}
    x_i^3 & x_i^6 & x_i & x_i^9 \\
    x_i^3 & x_i^2 & x_i & 1 \\
    \hline
    x_i^3 & x_i^2 & x_i & 1
\end{vmatrix} =
\begin{vmatrix}
    x_i^9 & x_i^6 & x_i^3 & x_i \\
    x_i^3 & x_i^2 & x_i & 1
\end{vmatrix}
\]

where just the \( i \)th row of each determinant has been displayed

Alternatively, one can proceed iteratively using the previous identities

\[
s_{0409}(x) = - s_{6151}(x) = + s_{6421}(x)
\]
Diagrammatically

Ex: $s_\kappa(x) = s_{0409}(x) = -s_{6151}(x) = +s_{6421}(x) = +s_\lambda(x)$
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Diagrammatically

**Ex:** \( s_\kappa(x) = s_{0409}(x) = -s_{6151}(x) = +s_{6421}(x) = +s_\lambda(x) \)

Note \( \lambda = (6421) = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix} \)
Frobenius notation and modifications

- Let \( \kappa_j = 0 \) unless \( j \in \{b_1+1, b_2+1, \ldots, b_r+1\} \)
- Let \( b_1 > b_2 > \cdots > b_r \geq 0 \) without loss of generality
- Let \( \kappa(j) = a_k + b_k + 1 \) if \( j = b_k+1 \) so that
  \[
  \kappa = (0^{b_r}, a_r+b_r+1, 0^{b_{r-1}-b_r-1}, \ldots, a_2+b_2+1, 0^{b_1-b_2-1}, a_1+b_1+1)
  \]
- Then, if \( a_1 > a_2 > \cdots > a_r \geq 0 \),
  \[
  s_{\kappa}(x) = (-1)^{b_1+b_2+\cdots+b_r} s_{\lambda}(x)
  \]
  with
  \[
  \lambda = \begin{pmatrix}
  a_1 & a_2 & \cdots & a_r \\
  b_1 & b_2 & \cdots & b_r
  \end{pmatrix}
  \]
  and \( r = r(\lambda) \)
Example

- For $\kappa = (0, 4, 0, 9)$ we have $\kappa = 0$ unless $j \in \{2, 4\}$
- Hence $r = 2$, $b_1 = 3$, $b_2 = 1$, with $b_1 > b_2 \geq 0$
- Since $\kappa_4 = a_1 + b_1 + 1 = 9$ and $\kappa_2 = a_2 + b_2 + 1 = 4$ we have $a_1 = 5$, $a_2 = 2$ with $a_1 > a_2 \geq 0$
- Hence we have $s_\kappa(x) = s_{0409}(x) = (-1)^{3+1} s_{6421}(x)$
- In Frobenius notation $\lambda = (6, 4, 2, 1) = \begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$
Example

- For $\kappa = (0, 4, 0, 9)$ we have $\kappa = 0$ unless $j \in \{2, 4\}$
- Hence $r = 2$, $b_1 = 3$, $b_2 = 1$, with $b_1 > b_2 \geq 0$
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- In Frobenius notation $\lambda = (6, 4, 2, 1) = \left(\begin{array}{cc} 5 & 2 \\ 3 & 1 \end{array}\right)$
Schur function series

Littlewood [1940] For all \( n \geq 1 \) and \( x = (x_1, x_2, \ldots, x_n) \):

\[
\sum_{\lambda} s_{\lambda}(x) = \prod_{1 \leq i \leq n} (1 - x_i)^{-1} \prod_{1 \leq j < k \leq n} (1 - x_j x_k)^{-1}
\]

\[
\sum_{\lambda \text{ even}} s_{\lambda}(x) = \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)^{-1}
\]

\[
\sum_{\lambda' \text{ even}} s_{\lambda}(x) = \prod_{1 \leq j < k \leq n} (1 - x_j x_k)^{-1}
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\]

\[
\sum_{\lambda \text{ even}} s_{\lambda}(x) = \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)^{-1}
\]

\[
\sum_{\lambda' \text{ even}} s_{\lambda}(x) = \prod_{1 \leq j < k \leq n} (1 - x_j x_k)^{-1}
\]

A partition is even if all its non-zero parts are even.

The infinite sums over \( \lambda \) involve no restriction on either \( \ell(\lambda) \) or \( \ell(\lambda') \), but \( s_{\lambda}(x) = 0 \) if \( \ell(\lambda) > n \).
Inverse Schur function series

Littlewood [1940]  For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$
\sum_{\lambda \in \mathcal{P}_0} (-1)^{(|\lambda| + r(\lambda))/2} s_{\lambda}(x) = \prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)
$$

$$
\sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2} s_{\lambda}(x) = \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)
$$

$$
\sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2} s_{\lambda}(x) = \prod_{1 \leq j < k \leq n} (1 - x_j x_k)
$$
Inverse Schur function series

- Littlewood [1940] For all \( n \geq 1 \) and \( x = (x_1, x_2, \ldots, x_n) \)

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\sum_{\lambda \in \mathcal{P}_0} (-1)^{|\lambda|+r(\lambda)}/2 s_{\lambda}(x) = \prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_j x_k)
\]

\[
\sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2} s_{\lambda}(x) = \prod_{1 \leq j \leq k \leq n} (1 - x_j x_k)
\]

\[
\sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2} s_{\lambda}(x) = \prod_{1 \leq j < k \leq n} (1 - x_j x_k)
\]

- These series are finite for all finite \( n \)

- For finite \( n \) both \( \ell(\lambda) \) and \( \ell(\lambda') \) are restricted, since for \( \lambda \in \mathcal{P}_t \) these differ by \( t \)
Determinantal identities

Littlewood [1940] For all \( n \geq 1 \) and \( x = (x_1, x_2, \ldots, x_n) \)

\[
\begin{align*}
\frac{|x_i^{n-j} - x_i^{n+j-1}|}{|x_i^{n-j}|} &= \sum_{\lambda \in \mathcal{P}_0} (-1)^{[|\lambda|+r(\lambda)]/2} s_\lambda(x) \\
\frac{|x_i^{n-j} - x_i^{n+j}|}{|x_i^{n-j}|} &= \sum_{\lambda \in \mathcal{P}_1} (-1)^{|\lambda|/2} s_\lambda(x) \\
\frac{|x_i^{n-j} + \chi_{j>1} x_i^{n+j-2}|}{|x_i^{n-j}|} &= \sum_{\lambda \in \mathcal{P}_{-1}} (-1)^{|\lambda|/2} s_\lambda(x)
\end{align*}
\]

the determinants are all \( n \times n \) with \( i, j = 1, 2, \ldots, n \)

and, for any proposition \( P \), \( \chi_P = \begin{cases} 1 & \text{if } P \text{ is true} \\ 0 & \text{if } P \text{ is false} \end{cases} \)
Lemma K [2008]  For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\frac{|x_i^{n-j} + q \chi_{j>-t} x_i^{n+t+j-1}|}{|x_i^{n-j}|} = \sum_{\lambda \in \mathcal{P}_t} (-1)^{|\lambda| - r(\lambda)(t+1)/2} q^r(\lambda) s_\lambda(x)$$

where $t$ is any integer, and $q$ is arbitrary

and the determinants are all $n \times n$

so that $i, j = 1, 2, \ldots, n$
General determinantal identity

**Lemma K [2008]** For all $n \geq 1$ and $x = (x_1, x_2, \ldots, x_n)$

$$\frac{|x_i^{n-j} + q \chi_{j \geq t} x_i^{n+t+j-1}|}{|x_i^{n-j}|} = \sum_{\lambda \in \mathcal{P}_t} (-1)^{|\lambda|-r(\lambda)(t+1)}/2 q^{r(\lambda)} s_{\lambda}(x)$$

where $t$ is any integer, and $q$ is arbitrary.

and the determinants are all $n \times n$.

so that $i, j = 1, 2, \ldots, n$.

The special cases:

$q = -1, t = 0$; $q = -1, t = 1$; $q = 1, t = -1$,
correspond to Littlewood’s previous formulae.
Algebraic proof

\[
\frac{|x_i^{n-j} + q \chi_{j>_{-t}} x_i^{n+t+j-1}|}{|x_i^{n-j}|} = \frac{|x_i^{n-j} + q \chi_{j>_{-t}} x_i^{2j-1+t+n-j}|}{|x_i^{n-j}|}
\]

\[
= \sum_{r=0}^{n} \sum_{\kappa} q^r s_\kappa(x) = \sum_{\lambda \in \mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)} q^r s_\lambda(x)
\]
Algebraic proof

\[
\frac{|x_i^{n-j} + q X_{j>_t} x_i^{2j-1+t+n-j}|}{|x_i^{n-j}|} = \frac{|x_i^{n-j} + q X_{j>_t} x_i^{2j-1+t+n-j}|}{|x_i^{n-j}|}
\]

\[
= \sum_{r=0}^{n} \sum_{\kappa} q^r s_{\kappa}(x) = \sum_{\lambda \in \mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)} q^r s_{\lambda}(x)
\]

- \( \kappa_j = 2j-1+t \) for \( j \in \{j_1, j_2, \ldots, j_r\} \) and \( \kappa_j = 0 \) otherwise
- with \( n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi t < 0 \)
- \( \lambda = \begin{pmatrix} j_1 - 1 + t & j_2 - 1 + t & \cdots & j_r - 1 + t \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \in \mathcal{P}_t \)
- \( r = r(\lambda) \)
- \( |\lambda| = 2((j_1-1) + (j_2-1) + \cdots + (j_r-1)) + r(t + 1) \)
Example with $n = 4$ and $t = 2$

\[
\begin{vmatrix}
  x_i^{4-j} + q \chi_{j>-2} x_i^{5+j} \\
  x_i^{4-j}
\end{vmatrix}
\]

\[
\begin{vmatrix}
  x_i^3 + q x_i^6 & x_i^2 + q x_i^7 & x_i + q x_i^8 & 1 + q x_i^9 \\
  x_i^3 & x_i^2 & x_i & 1
\end{vmatrix}
\]

\[
= s_{0000} + q \left( s_{3000} + s_{0500} + s_{0070} + s_{0009} \right) \\
+ q^2 \left( s_{3500} + s_{3070} + s_{0570} + s_{3009} + s_{0509} + s_{0079} \right) \\
+ q^3 \left( s_{3570} + s_{3509} + s_{3079} + s_{0579} \right) + q^4 s_{3579}
\]

\[
= 1 + q \left( s_3 - s_{41} + s_{511} - s_{6111} \right) \\
+ q^2 \left( -s_{44} + s_{541} - s_{552} - s_{6411} + s_{6521} - s_{6622} \right) \\
+ q^3 \left( -s_{555} + s_{6551} - s_{6652} + s_{6663} \right) + q^4 s_{6666}
\]
Example with \( n = 4 \) and \( t = 2 \) contd.

In Frobenius notation \( s_{\lambda}(x) = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \), we have

\[
1 + q \left[ \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right]
\]

\[+ q^2 \left[ - \begin{pmatrix} 32 \\ 10 \end{pmatrix} + \begin{pmatrix} 42 \\ 20 \end{pmatrix} - \begin{pmatrix} 43 \\ 21 \end{pmatrix} - \begin{pmatrix} 52 \\ 30 \end{pmatrix} + \begin{pmatrix} 53 \\ 31 \end{pmatrix} - \begin{pmatrix} 54 \\ 32 \end{pmatrix} \right] \]

\[+ q^3 \left[ - \begin{pmatrix} 432 \\ 210 \end{pmatrix} + \begin{pmatrix} 532 \\ 310 \end{pmatrix} - \begin{pmatrix} 542 \\ 320 \end{pmatrix} + \begin{pmatrix} 543 \\ 321 \end{pmatrix} \right] \]

\[+ q^4 \begin{pmatrix} 5432 \\ 3210 \end{pmatrix} \]
Example with $n = 4$ and $t = -2$

\[
\begin{array}{c}
\left| \begin{array}{c}
x_i^{4-j} + q \chi_{j>2} x_i^{1+j} \\
x_i^{4-j}
\end{array} \right|
\end{array}
\]

\[
= \left| \begin{array}{cccc}
x_i^3 & x_i^2 & x_i & 1 + q x_i^5 \\
x_i^3 & x_i^2 & x_i & 1
\end{array} \right|
\]

\[
= s_{0000} + q (s_{0030} + s_{0005}) + q^2 s_{0035}
\]

\[
= 1 + q (s_{111} - s_{2111}) - q^2 s_{2222}
\]

\[
= 1 + q \begin{pmatrix} 0 \\ 2 \end{pmatrix} - q \begin{pmatrix} 1 \\ 3 \end{pmatrix} - q^2 \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}
\]
Row length restricted Schur function series

\[ \sum_{\lambda: \ell(\lambda) \leq p} s_\lambda(x) \quad \text{with} \quad x = (x_1, x_2, \ldots, x_n), \quad n \geq 1, \quad p \geq 0 \]

\[ = \frac{|x_i^{n-j} - x_i^{n+p+j-1}|}{\prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(x_j - x_k)(1 - x_j x_k)} \quad \text{Macdonald} \]

\[ = \frac{|x_i^{n-j} - x_i^{n+p+j-1}|}{\prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(1 - x_j x_k)} / |x_i^{n-j}| \quad \text{Vandermonde} \]

\[ = \sum_{\mu \in \mathcal{P}_p} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_\mu(x) \frac{1}{\prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(1 - x_j x_k)} \quad \text{Lemma: } q=-1, t=p \]

\[ = \sum_{\mu \in \mathcal{P}_p} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_\mu(x) \frac{1}{\sum_{\nu \in \mathcal{P}_0} (-1)^{[|\nu| + r(\nu)]/2} s_\nu(x)} \quad \text{Littlewood} \]
Column length restricted Schur function series

- Using the conjugacy involution \( \omega : s_{\lambda}(x) \mapsto s_{\lambda'}(x) \) for all \( \lambda \)
- and noting that \( \lambda \in \mathcal{P}_t \implies \lambda' \in \mathcal{P}_{-t} \) for all \( t \), we have
Column length restricted Schur function series

Using the conjugacy involution \( \omega : s_{\lambda}(x) \mapsto s_{\lambda'}(x) \) for all \( \lambda \)

and noting that \( \lambda \in \mathcal{P}_t \implies \lambda' \in \mathcal{P}_{-t} \) for all \( t \), we have

\[
\sum_{\lambda : \ell(\lambda) \leq p} s_{\lambda}(x) \quad \text{with} \quad x = (x_1, x_2, \ldots, x_n), \; n \geq 1, \; p \geq 0
\]

\[
= \sum_{\mu \in \mathcal{P}_p} (-1)^{[\mu] - r(\mu)(p-1)/2} s_{\mu}(x)
\]

Conjugacy

\[
= \sum_{\nu \in \mathcal{P}_0} (-1)^{[\nu] + r(\nu)}/2 s_{\nu}(x)
\]

Van der Jeugt

\[
= \sum_{\mu \in \mathcal{P}_p} (-1)^{[\mu] - r(\mu)(p-1)/2} s_{\mu}(x)
\]

Lemma \( q = -(-1)^p \)

\[
t = -p
\]
So far

- We have recast the numerator of Macdonald’s formula as a signed sum of Schur functions
- We have then used conjugacy to prove Van der Jeugt’s conjecture
So far

- We have recast the numerator of Macdonald’s formula as a signed sum of Schur functions
- We have then used conjugacy to prove Van der Jeugt’s conjecture
- We have not exploited all of Littlewood’s series
- We have only used two special cases of the Lemma: $q = -1$, $t = p$ and $q = -(-1)^p$, $t = -p$
- But there exist further row (and as we shall see column) restricted Schur function series
Theorem [Macdonald 79; Désarménien 87, Stembridge 90, Proctor 90; Bressoud 98, Okada 98]

For all $n \geq 1$, $x = (x_1, x_2, \ldots, x_n)$ and $p \geq 0$:

$$\sum_{\lambda \in \lambda \leq p} s_{\lambda}(x) = \frac{|x_i^{n-j} - x_i^{n+p+j-1}|}{|x_i^{n-j}| \prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(1 - x_j x_k)}$$

$$\sum_{\lambda \text{ even : } \ell(\lambda) \leq 2p} s_{\lambda}(x) = \frac{|x_i^{n-j} - x_i^{n+2p+j}|}{|x_i^{n-j}| \prod_{1 \leq j \leq k \leq n}(1 - x_j x_k)}$$

$$\sum_{\lambda' \text{ even : } \ell(\lambda') \leq p} s_{\lambda}(x) = \frac{1}{2} \frac{|x_i^{n-j} - x_i^{n+p+j-2}| + \frac{1}{2} |x_i^{n-j} + x_i^{n+p+j-2}|}{|x_i^{n-j}| \prod_{1 \leq j < k \leq n}(1 - x_j x_k)}$$
Row length restricted Schur function series

Using the Lemma for given \( q \) and \( t \) as indicated, we find

**Corollary**

For all \( x = (x_1, x_2, \ldots) \)

\[ q = -1, t = p \]

\[ 
\sum_{\lambda: \ell(\lambda') \leq p} s_{\lambda}(x) = \frac{\sum_{\mu \in \mathcal{P}_p} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_{\mu}(x)}{\prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(1 - x_j x_k)}
\]

\[ q = -1, t = 2p + 1 \]

\[ 
\sum_{\lambda \text{ even}: \ell(\lambda') \leq 2p} s_{\lambda}(x) = \frac{\sum_{\mu \in \mathcal{P}_{2p+1}} (-1)^{[|\mu| - r(\mu)(2p)]/2} s_{\mu}(x)}{\prod_{1 \leq j < k \leq n}(1 - x_j x_k)}
\]

\[ q = \pm 1, t = p - 1 \]

\[ 
\sum_{\lambda': \ell(\lambda') \leq p} s_{\lambda}(x) = \frac{\sum_{\mu \in \mathcal{P}_{p-1}: r(\mu) \text{ even}} (-1)^{[|\mu| - r(\mu)p]/2} s_{\mu}(x)}{\prod_{1 \leq j < k \leq n}(1 - x_j x_k)}
\]
Row length restricted Schur function series

Littlewood’s inverse Schur function series formulae then give:

Corollary For all $x = (x_1, x_2, \ldots )$

$q = -1, t = p$

$$
\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) = \frac{\sum_{\mu \in \mathcal{P}_p} (-1)^{|\mu| - r(\mu)(p-1)} / 2 s_\mu(x)}{\sum_{\nu \in \mathcal{P}_0} (-1)^{|\nu| + r(\nu)} / 2 s_\nu(x)}
$$

$q = -1, t = 2p + 1$

$$
\sum_{\lambda \text{ even} : \ell(\lambda') \leq 2p} s_\lambda(x) = \frac{\sum_{\mu \in \mathcal{P}_{2p+1}} (-1)^{|\mu| - r(\mu)(2p)} / 2 s_\mu(x)}{\sum_{\nu \in \mathcal{P}_1} (-1)^{|\nu| / 2} s_\nu(x)}
$$

$q = \pm 1, t = p - 1$

$$
\sum_{\lambda' \text{ even} : \ell(\lambda') \leq p} s_\lambda(x) = \frac{\sum_{\mu \in \mathcal{P}_{p-1} : r(\mu) \text{ even}} (-1)^{|\mu| - r(\mu) p} / 2 s_\mu(x)}{\sum_{\nu \in \mathcal{P}_{p-1}} (-1)^{|\nu| / 2} s_\nu(x)}
$$
Column length restricted Schur function series

Using the involution \( \omega : s_\lambda(x) \mapsto s_{\lambda'}(x) \) for all \( \lambda \)

and noting that \( \lambda \in \mathcal{P}_t \implies \lambda' \in \mathcal{P}_{-t} \) for all \( t \), we have

Corollary

For all \( x = (x_1, x_2, \ldots) \)

\[
\sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x) = \frac{\sum_{\mu \in \mathcal{P}_-p} (-1)^{|\mu| - r(\mu)(p-1)/2} s_\mu(x)}{\sum_{\nu \in \mathcal{P}_0} (-1)^{|\nu| + r(\nu)/2} s_\nu(x)}
\]

\[
\sum_{\lambda' \text{ even} : \ell(\lambda) \leq 2p} s_\lambda(x) = \frac{\sum_{\mu \in \mathcal{P}_{-2p-1}} (-1)^{|\mu| - r(\mu)(2p)/2} s_\mu(x)}{\sum_{\nu \in \mathcal{P}_{-1}} (-1)^{|\nu|/2} s_\nu(x)}
\]

\[
\sum_{\lambda \text{ even} : \ell(\lambda) \leq p} s_\lambda(x) = \frac{\sum_{\mu \in \mathcal{P}_{-p+1} : r(\mu) \text{ even}} (-1)^{|\mu| - r(\mu)p/2} s_\mu(x)}{\sum_{\nu \in \mathcal{P}_1} (-1)^{|\nu|/2} s_\nu(x)}
\]
Littlewood’s inverse Schur function series formulae then give:

**Corollary**  For all $x = (x_1, x_2, \ldots)$

$$\sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x) = \sum_{\mu \in \mathcal{P}_{-p}} (-1)^{|\mu| - r(\mu)(p-1)}/2 \frac{s_\mu(x)}{\prod_{1 \leq i \leq n}(1 - x_i) \prod_{1 \leq j < k \leq n}(1 - x_j x_k)}$$

$$\sum_{\lambda' \text{ even} : \ell(\lambda) \leq 2p} s_\lambda(x) = \sum_{\mu \in \mathcal{P}_{-2p-1}} (-1)^{|\mu| - r(\mu)(2p)}/2 \frac{s_\mu(x)}{\prod_{1 \leq j < k \leq n}(1 - x_j x_k)}$$

$$\sum_{\lambda \text{ even} : \ell(\lambda) \leq p} s_\lambda(x) = \sum_{\mu \in \mathcal{P}_{-p+1}} (-1)^{|\mu| - r(\mu)p}/2 \frac{s_\mu(x)}{\prod_{1 \leq j \leq k \leq n}(1 - x_j x_k)}$$

**Note:** The first of these was Van der Jeugt’s Conjecture
Column length restricted Schur function series

Using \((q, t) = (-(−1)^p, −p), (±1, −p + 1)\) and \((1, −2p − 1)\) in our Lemma, we find

**Theorem** For all \(n \geq 1, \ x = (x_1, x_2, \ldots, x_n)\) and \(p \geq 0:\)

\[
\sum_{\lambda: \ell(\lambda) \leq p} s_\lambda(x) = \frac{|x_i^{n-j} - (-1)^p \chi_{j \geq p} x_i^{n-p+j-1}|}{|x_i^{n-j}| \prod_{1 \leq i \leq n} (1 - x_i) \prod_{1 \leq j < k \leq n} (1 - x_jx_k)}
\]

\[
\sum_{\lambda \text{ even}: \ell(\lambda) \leq p} s_\lambda(x) = \frac{\frac{1}{2} |x_i^{n-j} - \chi_{j \geq p} x_i^{n-p+j}| + \frac{1}{2} |x_i^{n-j} + \chi_{j \geq p} x_i^{n-p+j}|}{|x_i^{n-j}| \prod_{1 \leq j < k \leq n} (1 - x_jx_k)}
\]

\[
\sum_{\lambda' \text{ even}: \ell(\lambda) \leq 2p} s_\lambda(x) = \frac{|x_i^{n-j} + \chi_{j \geq 2p+1} x_i^{n-2p+j-2}|}{|x_i^{n-j}| \prod_{1 \leq j < k \leq n} (1 - x_jx_k)}
\]
Row length restricted Schur function series

Alternative universal expressions giving each restricted series as a product of an unrestricted series and a correction factor for all \( x = (x_1, x_2, \ldots) \) take the form

\[
\sum_{\lambda: \ell(\lambda') \leq p} s_{\lambda}(x) = \sum_{\lambda} s_{\lambda}(x) \cdot \sum_{\mu \in \mathcal{P}_p} (-1)^{|\mu| - r(\mu)(p-1)/2} s_{\mu}(x)
\]

\[
\sum_{\lambda \text{ even} : \ell(\lambda') \leq 2p} s_{\lambda}(x) = \sum_{\lambda \text{ even}} s_{\lambda}(x) \cdot \sum_{\mu \in \mathcal{P}_{2p+1}} (-1)^{|\mu| - r(\mu)(2p)/2} s_{\mu}(x)
\]

\[
\sum_{\lambda' \text{ even} : \ell(\lambda') \leq p} s_{\lambda}(x) = \sum_{\lambda' \text{ even}} s_{\lambda}(x) \cdot \sum_{\mu \in \mathcal{P}_{p-1}: r(\mu) \text{ even}} (-1)^{|\mu| - r(\mu)p/2} s_{\mu}(x)
\]
Column length restricted Schur function series

Alternative universal expressions giving each restricted series as a product of an unrestricted series and a correction factor for all $x = (x_1, x_2, \ldots)$ take the form

$$\sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{-p}} (-1)^{[|\mu|-r(\mu)(p-1)]/2} s_\mu(x)$$

$$\sum_{\lambda' \text{ even} : \ell(\lambda) \leq 2p} s_\lambda(x) = \sum_{\lambda' \text{ even}} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{-2p-1}} (-1)^{[|\mu|-r(\mu)(2p)]/2} s_\mu(x)$$

$$\sum_{\lambda \text{ even} : \ell(\lambda) \leq p} s_\lambda(x) = \sum_{\lambda \text{ even}} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{-p+1 : r(\mu) \text{ even}}} (-1)^{[|\mu|-r(\mu)p]/2} s_\mu(x)$$
Row length restricted Schur function series

Alternative universal expressions giving each restricted series as a product of an unrestricted series and a correction factor for all \( x = (x_1, x_2, \ldots) \) take the form

\[
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\]

\[
\sum_{\lambda \text{ even} : \ell(\lambda') \leq 2p} s_\lambda(x) = \sum_{\lambda \text{ even}} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{2p+1}} (-1)^{|\mu| - r(\mu)(2p)} s_\mu(x)
\]

\[
\sum_{\lambda' \text{ even} : \ell(\lambda') \leq p} s_\lambda(x) = \sum_{\lambda' \text{ even}} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{p-1} : r(\mu) \text{ even}} (-1)^{|\mu| - r(\mu)p} s_\mu(x)
\]
The row length restricted series takes the form

$$\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) = \sum_\lambda s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_p} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_\mu(x)$$

The column length restricted series takes the form

$$\sum_{\lambda: \ell(\lambda) \leq p} s_\lambda(x) = \sum_\lambda s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{-p}} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_\mu(x)$$
**Rank restricted Schur function series**

- The **row** length restricted series takes the form

\[
\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_p} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_\mu(x)
\]

- The **column** length restricted series takes the form

\[
\sum_{\lambda: \ell(\lambda) \leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_{-p}} (-1)^{[|\mu| - r(\mu)(p-1)]/2} s_\mu(x)
\]

- **Conjecture** The **rank** restricted series takes the form

\[
\sum_{\lambda: r(\lambda) \leq p} s_\lambda(x) = \sum_{\lambda} s_\lambda(x) \cdot \sum_{\mu \in \mathcal{P}_0: r(\mu) = p+1} (-1)^{[|\mu| + r(\mu)]/2} s_\mu(x)
\]
So far

We have obtained three determinantal formulae for column length restricted partitions analogous to those for row length restricted partitions.
So far

- We have obtained three determinantal formulae for column length restricted partitions analogous to those for row length restricted partitions.
- We have not explained why the various determinants lead to row or column length restrictions.
- To do this we need to exploit the fact that they define characters of particular representations of classical groups as emphasized by Okada.
- Then we may look for an alternative way of evaluating these characters through the use of Howe dual pairs of groups.
Classical groups and their characters

Let \( x = (x_1, x_2, \ldots, x_n) \) and \( \overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n) \) with \( x_i = e^{\epsilon_i} \) and \( \overline{x}_i = x_i^{-1} = e^{-\epsilon_i} \) for \( i = 1, 2, \ldots, n \)

\[
\text{ch} \ V_\lambda^{GL(n)} = \left| \begin{array}{c} \lambda_j + n - j \\ x_i \\ x_i^{n-j} \end{array} \right| \\
\text{ch} \ V_\lambda^{SO(2n+1)} = \left| \begin{array}{c} \lambda_j + n - j + \frac{1}{2} \\ x_i \\ x_i^{n-j+\frac{1}{2}} \end{array} \right| - \left| \begin{array}{c} \lambda_j + n - j - \frac{1}{2} \\ x_i^{-1} \\ x_i^{n-j-\frac{1}{2}} \end{array} \right| \\
\text{ch} \ V_\lambda^{Sp(2n)} = \left| \begin{array}{c} \lambda_j + n - j + 1 \\ x_i \\ x_i^{n-j+1} \end{array} \right| - \left| \begin{array}{c} \lambda_j + n - j - 1 \\ x_i^{-1} \\ x_i^{n-j-1} \end{array} \right| \\
\text{ch} \ V_\lambda^{SO(2n)} = \left| \begin{array}{c} x_i^{n-j} + \overline{x}_i^{n-j} \\ x_i \\ x_i^{n-j} \end{array} \right| + \left| \begin{array}{c} \lambda_j + n - j \\ x_i \\ \lambda_j + n - j \end{array} \right| - \left| \begin{array}{c} \lambda_j + n - j \\ \overline{x}_i \\ \lambda_j + n - j \end{array} \right|
Characters expressed in terms of Schur functions

\[ \text{ch } V_{GL(n)}^\lambda = s_\lambda(x) \]

\[ \text{ch } V_{SO(2n+1)}^\lambda = \sum_{\mu \in \mathcal{P}_0} (-1)^{|\mu|-r(\mu)} \frac{s_{\lambda/\mu}(x, \bar{x})}{2} \]

\[ \text{ch } V_{SO(2n+1)}^{\lambda+\frac{1}{2}} = \text{ch } V_{SO(2n)}^\Delta \sum_{\mu \in \mathcal{P}_{-1}} (-1)^{|\mu|/2} s_{\lambda/\mu}(x, \bar{x}) \]

\[ \text{ch } V_{Sp(2n)}^\lambda = \sum_{\mu \in \mathcal{P}_{-1}} (-1)^{|\mu|/2} s_{\lambda/\mu}(x, \bar{x}) \]

\[ \text{ch } V_{SO(2n)}^\lambda = \sum_{\mu \in \mathcal{P}_{1}} (-1)^{|\mu|/2} s_{\lambda/\mu}(x, \bar{x}) \]

\[ \text{ch } V_{SO(2n)}^{\lambda+\frac{1}{2}} = \text{ch } V_{SO(2n)}^\Delta \sum_{\mu \in \mathcal{P}_0} (-1)^{|\mu|+r(\mu)}/2 s_{\lambda/\mu}(x, \bar{x}) \]
Row length restricted series and characters

**Theorem** [Macdonald, Désarménien, Stembridge, Proctor, Bressoud, Okada]

\[
\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) = \frac{|x_i^{n-j} - x_i^{n+p+j-1}|}{|x_i^{n-j} - x_i^{n+j-1}|} = x^{p/2} \operatorname{ch} V_{SO(2n+1)}^{(p/2)^n}(x, \overline{x}, 1)
\]

\[
\sum_{\lambda \text{ even}: \ell(\lambda') \leq 2p} s_\lambda(x) = \frac{|x_i^{n-j} - x_i^{n+2p+j}|}{|x_i^{n-j} - x_i^{n+j}|} = x^p \operatorname{ch} V_{Sp(2n)}^{p^n}(x, \overline{x})
\]

\[
\sum_{\lambda' \text{ even}: \ell(\lambda') \leq p} s_\lambda(x) = \frac{|x_i^{n-j} - x_i^{n+p+j-2}| + |x_i^{n-j} + x_i^{n+p+j-2}|}{|x_i^{n-j} + x_i^{n+j-2}|}
\]

\[= x^{p/2} \operatorname{ch} V_{SO(2n)}^{(p/2)^{n-1}, (-)^n(p/2)}(x, \overline{x})
\]

where \(x = x_1x_2\cdots x_n = \operatorname{ch} V_{GL(n)}^1(x)\)
Proof of formulae in terms of characters

- Start from the original determinantal formulae
- In each determinant permute columns under \( j \rightarrow n - j + 1 \)
- Extract factors \((-1)^n\) by changing signs of all terms of the form \( x_i^a - x_i^b \)
- Extract factors
  - \( x_i^{n-1/2 + p/2} \) and \( x_i^{n-1/2} \)
  - \( x_i^{n+p} \) and \( x_i^n \)
  - \( x_i^{n-1 + p/2} \) and \( x_i^{n-1} \)

from each row of numerator and denominator determinants
Howe dual pairs of groups

Definition [Howe 85]

Let groups $G$ and $H$ act on a linear vector space $V$.

Let their actions mutually commute.

As a representation of $G \times H$, let

$$V = \bigoplus_{k \in K} V_G^{\lambda(k)} \otimes V_H^{\mu(k)}$$

$k$ varies over some index set $K$.

$V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ are irreps of $G$ and $H$.

$V_G^{\lambda(k)}$ and $V_H^{\mu(k)}$ vary without repetition.

In such a case we say that $G$ and $H$ form a (Howe) dual pair with respect to $V$. 
Howe dual pairs of classical groups

- In some cases, $V$ is an irrep of a group $F \supseteq G \times H$

- On restriction to the subgroup $G \times H$

$$
\text{ch } V^F_{G \times H} = \sum_{k \in K} \text{ch } V^\lambda_{G}(k) \text{ch } V^\mu_{H}(k)
$$
Howe dual pairs of classical groups

- In some cases $V$ is an irrep of a group $F \supseteq G \times H$
- On restriction to the subgroup $G \times H$

$$\text{ch } V^F_{G \times H} = \sum_{k \in K} \text{ch } V_{G}^{\lambda(k)} \text{ch } V_{H}^{\mu(k)}$$

Ex: [Howe 89, Hasegawa 89] For $V$ the spin irrep of an orthogonal group with character $\text{ch } V^\Delta$, dual pairs are defined through each of the following restrictions:

- $O(4np) \supseteq SO(2n) \times O(2p)$
- $O(4np + 2p) \supseteq SO(2n + 1) \times O(2p)$
- $O(4np + 2n) \supseteq SO(2n) \times O(2p + 1)$
- $O(4np + 2n + 2p + 1) \supseteq SO(2n + 1) \times O(2p + 1)$
- $O(4np) \supseteq Sp(2n) \times Sp(2p)$
Notation for $p^n$-complements

- For any partition $\lambda \subseteq n^p$ we have $\lambda' \subseteq p^n$

- In such a case, let $\lambda^\dagger = (p - \lambda'_n, \ldots, p - \lambda'_2, p - \lambda'_1)$

- Then $\lambda^\dagger$ is also a partition
Notation for $p^n$-complements

- For any partition $\lambda \subseteq n^p$ we have $\lambda' \subseteq p^n$
- In such a case, let $\lambda^\dagger = (p - \lambda'_n, \ldots, p - \lambda'_2, p - \lambda'_1)$
- Then $\lambda^\dagger$ is also a partition

Ex: If $p = 4$, $n = 5$ and $\lambda = (4, 3, 1)$ then $\lambda' = (3, 2, 2, 1)$ and $\lambda^\dagger = (4, 3, 2, 2, 1)$

$$F^\lambda = \begin{array}{c|c|c|c|c|c|c|c} & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \end{array}$$

$$F^{\lambda'} = \begin{array}{c|c|c|c|c|c|c|c} & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \hline & & & & & & & \\ \end{array}$$

$$F^{\lambda^\dagger} = \begin{array}{c|c|c|c|c|c|c|c|c|c|c} & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \hline & & & & & & & & & & \\ \end{array}$$

- Note: $0^\dagger = p^n = (p, p, \ldots, p)$
The spin module and Howe dual pairs

**Theorem** [Morris 58,60; Hasegawa 89; Terada 93; Bump and Gamburd 05] On restriction to the appropriate subgroup:

\[
\begin{align*}
\text{ch } V_{O(4np)} & = \sum_{\lambda \subseteq n^p} \text{ch } V_{SO(2n)}^{\lambda \dagger} \text{ch } V_{O(2p)}^{\lambda} \\
\text{ch } V_{O(4np+2p)} & = \sum_{\lambda \subseteq n^p} \text{ch } V_{SO(2n+1)}^{\lambda \dagger} \text{ch } V_{O(2p)}^{\Delta;\lambda} \\
\text{ch } V_{O(4np+2n)} & = \sum_{\lambda \subseteq n^p} \text{ch } V_{SO(2n)}^{\Delta;\lambda \dagger} \text{ch } V_{O(2p+1)}^{\lambda} \\
\text{ch } V_{O(4np+2n+2p+1)} & = \sum_{\lambda \subseteq n^p} \text{ch } V_{SO(2n+1)}^{\Delta;\lambda \dagger} \text{ch } V_{O(2p+1)}^{\Delta;\lambda} \\
\text{ch } V_{O(4np)} & = \sum_{\lambda \subseteq n^p} \text{ch } V_{Sp(2n)}^{\lambda \dagger} \text{ch } V_{Sp(2p)}^{\lambda}
\end{align*}
\]
Exploitation of Howe duality

Let \((G, H)\) be a Howe dual pair with \(F \supseteq G \times H\) such that
\[
\text{ch} \, V^F_{G \times H} = \sum_{k \in K} \text{ch} \, V^\lambda(k)_G \text{ch} \, V^\mu(k)_H
\]

The character \(\text{ch} \, V^\lambda(k)_G\) is just the coefficient of \(\text{ch} \, V^\mu(k)_H\) in any formula we can devise for \(\text{ch} \, V^F_{G \times H}\).
Exploitation of Howe duality

Let \((G, H)\) be a Howe dual pair with \(F \supseteq G \times H\) such that

\[ \text{ch} V^F_{G \times H} = \sum_{k \in K} \text{ch} V^\lambda(k)_G \text{ch} V^\mu(k)_H \]

The character \(\text{ch} V^\lambda(k)_G\) is just the coefficient of \(\text{ch} V^\mu(k)_H\) in any formula we can devise for \(\text{ch} V^F_{G \times H}\).

In the case of the spin character identities all that is needed are:

- dual Cauchy formula
- expressions for classical group characters in terms of Schur functions [Littlewood 1940]
- some modification rules [Newell 1951]
Spin characters and their decomposition

In terms of appropriate parameters

$$\text{ch } V_{O(2n)}^\Delta(x, \bar{x}) = \prod_{i=1}^{n} \left( x_i^{\frac{1}{2}} + x_i^{-\frac{1}{2}} \right) = x^{-1} \prod_{i=1}^{n} (1 + x_i)$$

$$\text{ch } V_{O(4np)}^\Delta(xy, x\bar{y}, \bar{x}y, \bar{x}\bar{y})$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{p} \left( x_i^{\frac{1}{2}} y_j^{\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}} \right) \left( x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}} \right)$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{p} (x_i + \bar{x}_i + y_j + \bar{y}_j)$$

$$= x^{-p} \prod_{i=1}^{n} \prod_{j=1}^{p} (1 + x_i y_j)(1 + x_i \bar{y}_j) = x^{-p} \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x) s_{\zeta}(y, \bar{y})$$
Application to Howe dual pair contd.

\[
\begin{align*}
=x^-p \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x) s_\zeta(y, \bar{y})
&= x^-p \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x) \text{ch} V_\zeta^{GL(2p)}(y, \bar{y}) \\
&= x^-p \sum_{\zeta \subseteq n^{2p}} s_{\zeta'}(x) \sum_{\beta: \beta' \text{even}} \text{ch} V_{\zeta/\beta}^{\gamma}(y, \bar{y}) \\
&= x^-p \sum_{\eta \subseteq n^{2p}} W_{2p} \left( \sum_{\beta: \beta' \text{even}} s_{\eta'}(x) s_{\beta'}(x) \right) \text{ch} V_{\eta}^{\gamma}(y, \bar{y}) \\
&= x^-p \sum_{\eta \subseteq n^{2p}} W_{2p} \left( \sum_{\delta \text{ even}} s_{\eta'}(x) s_{\delta}(x) \right) \text{ch} V_{\eta}^{\gamma}(y, \bar{y}) \\
&= \sum_{\lambda \subseteq n^{p}} \text{ch} V_{\lambda}^{\gamma^{\dagger}}(x, \bar{x}) \text{ch} V_{\lambda}^{\gamma}(y, \bar{y}) \quad \text{dual pair Theorem}
\end{align*}
\]

where \( W_{2p} \) restricts any sum of Schur functions \( s_\nu(x) \) to those having \( \nu_1 = \ell(\nu') \leq 2p \)
It follows that

\[ \text{ch } V_{Sp(2n)}^{\lambda \dagger}(x, \bar{x}) = x^{-p} \sum_{\eta \subseteq n^{2p}} \varepsilon_{\eta, \lambda} \mathcal{W}_{2p} \left( \sum_{\delta \text{ even}} s_{\eta'}(x) s_{\delta}(x) \right) \]

where the modification rules for \( Sp(2p) \) characters are such that

\[ \varepsilon_{\eta, \lambda} = \begin{cases} 
\pm 1 & \text{if } \text{ch } V_{Sp(2p)}^{\eta}(y, \bar{y}) = \pm \text{ch } V_{Sp(2p)}^{\lambda}(y, \bar{y}) \\
0 & \text{otherwise}
\end{cases} \]
Character formula

It follows that

\[
\text{ch } V_{Sp(2n)}^{\lambda \dagger}(x, \bar{x}) = x^{-p} \sum_{\eta \subseteq n^{2p}} \varepsilon_{\eta, \lambda} \mathcal{W}_{2p}\left( \sum_{\delta \text{ even}} s_{\eta'}(x) s_{\delta}(x) \right)
\]

where the modification rules for \( Sp(2p) \) characters are such that

\[
\varepsilon_{\eta, \lambda} = \begin{cases} 
\pm 1 & \text{if } \text{ch } V_{Sp(2p)}^{\eta}(y, \bar{y}) = \pm \text{ch } V_{Sp(2p)}^{\lambda}(y, \bar{y}) \\
0 & \text{otherwise}
\end{cases}
\]

To be more precise [K and Wybourne 00]

\[
\text{ch } V_{Sp(2n)}^{\lambda \dagger}(x, \bar{x}) = x^{-p} \sum_{\alpha \in \mathcal{P}_{-1}} \sum_{\delta \text{ even}} (-1)^{|\alpha|/2} \mathcal{W}_{2p}\left( s_{(\lambda, \alpha)'}(x) s_{\delta}(x) \right)
\]

\[\text{SLC61-2008 – p. 47}\]
Character formula

Here \((\lambda, \alpha) = (\lambda_1, \ldots, \lambda_p, \alpha_1 \ldots, \alpha_p)\)

Standardisation is necessary if \(\lambda_p < \alpha_1\)

For given \(\lambda\) only a finite number of terms \(\alpha \in P_{-1}\) give non-zero contributions
Character formula

Here \((\lambda, \alpha) = (\lambda_1, \ldots, \lambda_p, \alpha_1 \ldots, \alpha_p)\)

Standardisation is necessary if \(\lambda_p < \alpha_1\)

For given \(\lambda\) only a finite number of terms \(\alpha \in \mathcal{P}_{-1}\) give non-zero contributions

Example

If \(\lambda = 0\) then only the case \(\alpha = 0\) survives. In this case \(\lambda^\dagger = (p^n)\) and

\[
\text{ch } V_{Sp(2n)}^p(x, \bar{x}) = x^{-p} \mathcal{W}_{2p} \left( \sum_{\delta \text{ even}} s_\delta(x) \right)
\]

\[
= x^{-p} \sum_{\delta \text{ even}, l(\delta') \leq 2p} s_\delta(x) \quad \text{as before}
\]
Character formula

- Howe duality thus leads directly to a formula for one of the row length restricted Schur function series
- It involves a character of rectangular shape, since $\lambda^\dagger = p^n$
Character formula

- Howe duality thus leads directly to a formula for one of the row length restricted Schur function series

- It involves a character of rectangular shape, since $\lambda^\dagger = p^n$

- If $\lambda = m$ then only the case $\alpha = 0$ survives. In this case $\lambda^\dagger = p^n/1^m = (p^{n-m}, (p-1)^m)$ and

$$\text{ch} \ V_{\text{Sp}(2n)}^{p^{n-m},(p-1)^m}(x, \overline{x}) = x^{-p} \left( \sum_{\delta \text{ even}} s_{1^m}^{m}(x) s_{\delta}(x) \right)$$

$$= x^{-p} \sum_{\mu \in (2p)^n : \text{oddrows}(\mu) = p} s_{\mu}(x)$$

- This is a formula for a character of near rectangular shape, previously derived by Krattenthaler [98]
Character formula

If $\lambda = 1^m$ then two terms survive. In this case $\lambda^\dagger = p^n/m = (p^{n-1}, p - m)$ and

$$\text{ch} V_{Sp(2n)}^{p^{n-1}, p-m}(x, \bar{x}) = x^{-p} \mathcal{W}_{2p} \left( \sum_{\delta \text{ even}} (s_m(x) - s_{2p+2-m}(x)) s_\delta(x) \right)$$

This gives another character of near rectangular shape.

Some care is required to effect the cancellations necessary to express the character as a sum of wholly positive terms, see [Krattenthaler 98]

Further examples can easily be generated, but they involve more complicated cancellations.
Character formula

If $\lambda = 1^m$ then two terms survive. In this case $\lambda^\dagger = p^n/m = (p^{n-1}, p - m)$ and

$$\text{ch } V_{Sp(2n)}^{p^{n-1},p-m}(x, \overline{x}) = x^{-p} \mathcal{W}_{2p}\left( \sum_{\delta \text{ even}} (s_m(x) - s_{2p+2-m}(x)) s_\delta(x) \right)$$

This gives another character of near rectangular shape.

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Further examples can easily be generated, but they involve more complicated cancellations.
The spin module and Howe dual pairs

Thus we have recovered the formula for the symplectic group characters as a sum of row length restricted Schur functions specified by even partitions.

Similar formulae for orthogonal group characters may be recovered in the same way using Howe dual pairs.

In each case the row length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $\lambda^\dagger$ and $\lambda$. 
The spin module and Howe dual pairs

Thus we have recovered the formula for the symplectic group characters as a sum of row length restricted Schur functions specified by even partitions.

Similar formulae for orthogonal group characters may be recovered in the same way using Howe dual pairs.

In each case the row length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $\lambda^\dagger$ and $\lambda$.

We would like to identify other Howe dual pairs that might lead to characters expressible as our sums of column length restricted Schur functions.

Such characters are necessarily infinite dimensional.
The metaplectic module and Howe dual pairs

- We need an infinite-dimensional analogue of the spin representation of the orthogonal group
- This is provided by the metaplectic representation of the symplectic group
The metaplectic module and Howe dual pairs

We need an infinite-dimensional analogue of the spin representation of the orthogonal group.

This is provided by the metaplectic representation of the symplectic group.

Ex: [Howe 89] For $V$ the metaplectic irrep of a symplectic group with character $\chi V^\Lambda$, dual pairs are defined through each of the following restrictions:

\[
\begin{align*}
Sp(4np) & \supseteq Sp(2n) \times O(2p) \\
Sp(4np + 2p) & \supseteq Sp(2n) \times O(2p + 1) \\
Sp(4np) & \supseteq SO(2n) \times Sp(2p)
\end{align*}
\]
Metaplectic dual pair character formula

**Theorem** [Moshinsky and Quesne 71, Kashiwara and Vergne 78, Howe 85, K and Wybourne 85]

On restriction to the appropriate subgroup:

\[
\text{ch } V_{Sp(4np)}^\Delta = \sum_{\lambda: \lambda_1' + \lambda_2' \leq 2p, \lambda_1' \leq n} \text{ch } V_{Sp(2n)}^p(\lambda) \text{ ch } V_{O(2p)}^\lambda
\]

\[
\text{ch } V_{Sp(4np+2n)}^\Delta = \sum_{\lambda: \lambda_1' + \lambda_2' \leq 2p+1, \lambda_1' \leq n} \text{ch } V_{Sp(2n)}^{p+\frac{1}{2}}(\lambda) \text{ ch } V_{O(2p+1)}^\lambda
\]

\[
\text{ch } V_{Sp(4np)}^\Delta = \sum_{\lambda: \lambda_1' \leq \min(p,n)} \text{ch } V_{SO(2n)}^p(\lambda) \text{ ch } V_{Sp(2p)}^\lambda
\]
In terms of appropriate parameters

\[ \text{ch } V_{Sp(2n)}(x, \bar{x}) = \prod_{i=1}^{n} (x_i^{-\frac{1}{2}} - x_i^{\frac{1}{2}})^{-1} = x \prod_{i=1}^{n} (1 - x_i)^{-1} \]

\[ \text{ch } V_{Sp(4np)}(xy, x\bar{y}, \bar{x}y, \bar{xy}) \]

\[ = \prod_{i=1}^{n} \prod_{j=1}^{p} (x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}} - x_i^{\frac{1}{2}} y_j^{\frac{1}{2}})^{-1} (x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}} - x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}})^{-1} \]

\[ = x^p \prod_{i=1}^{n} \prod_{j=1}^{p} (1 - x_iy_j)^{-1} (1 - x_i\bar{y}_j)^{-1} \]

\[ = x^p \sum_{\zeta: \ell(\zeta) \leq \min(n,2p)} s_{\zeta}(x) s_{\zeta}(y, \bar{y}) \]
Application to Howe dual pair contd.

\[= x^p \sum_{\zeta: \ell(\zeta) \leq \min(n,2p)} s_\zeta(x) \ s_\zeta(y, \bar{y})\]

\[= x^p \sum_{\zeta: \ell(\zeta) \leq \min(n,2p)} s_\zeta(x) \ \text{ch} \ V^\zeta_{GL(2p)}(y, \bar{y})\]

\[= x^p \sum_{\zeta: \ell(\zeta) \leq \min(n,2p)} s_\zeta(x) \ \sum_{\delta \text{ even}} \text{ch} \ V^{\zeta/\delta}_{O(2p)}(y, \bar{y})\]

\[= x^p \sum_{\eta: \ell(\eta) \leq \min(n,2p)} \mathcal{L}_{2p} \left( \sum_{\delta \text{ even}} s_\eta(x) \ s_\delta(x) \right) \ \text{ch} \ V^n_{O(2p)}(y, \bar{y})\]

\[= \sum_{\lambda: \lambda'_1 + \lambda'_2 \leq 2p, \lambda'_1 \leq n} \text{ch} \ V^{p(\lambda)}_{Sp(2n)}(x, \bar{x}) \ \text{ch} \ V^\lambda_{O(2p)}(y, \bar{y}) \quad \text{dual pair}\]

where \( \mathcal{L}_{2p} \) restricts any sum of Schur functions \( s_\nu(x) \) to those having \( \nu'_1 = \ell(\nu) \leq 2p \)
It follows that

\[
\text{ch } V_{Sp(2n)}^{p(\lambda)}(x, \overline{x}) = x^p \sum_{\eta: \ell(\zeta) \leq \min(n, 2p)} \varepsilon_{\eta, \lambda} \mathcal{L}_{2p}\left( \sum_{\delta \text{ even}} s_{\eta}(x) s_{\delta}(x) \right)
\]

where the modification rules for \( O(2p) \) characters are such that

\[
\varepsilon_{\eta, \lambda} = \begin{cases} 
\pm 1 & \text{if } \text{ch } V_{O(2p)}^{\eta}(y, \overline{y}) = \pm \text{ch } V_{O(2p)}^{\lambda}(y, \overline{y}) \\
0 & \text{otherwise}
\end{cases}
\]
Character formula

It follows that

\[ \text{ch} \ V_{Sp(2n)}^{p(\lambda)}(x, \bar{x}) = x^p \sum_{\eta: \ell(\zeta) \leq \min(n,2p)} \varepsilon_{\eta,\lambda} \mathcal{L}_{2p} \left( \sum_{\delta \text{ even}} s_{\eta}(x) s_{\delta}(x) \right) \]

where the modification rules for \( O(2p) \) characters are such that

\[ \varepsilon_{\eta,\lambda} = \begin{cases} 
\pm 1 & \text{if } \text{ch} \ V_{O(2p)}^{\eta}(y, \bar{y}) = \pm \text{ch} \ V_{O(2p)}^{\lambda}(y, \bar{y}) \\
0 & \text{otherwise} 
\end{cases} \]

In the special case \( \lambda = 0 \) this gives

\[ \text{ch} \ V_{Sp(2n)}^{p(0)}(x, \bar{x}) = x^p \mathcal{L}_{2p} \left( \sum_{\delta \text{ even}} s_{\delta}(x) \right) = x^p \sum_{\delta \text{ even}: \ell(\delta) \leq 2p} s_{\delta}(x) \]
The metaplectic module and Howe dual pairs

Thus we have obtained a formula for a particular symplectic group character as a sum of column length restricted Schur functions specified by even partitions.

Our other column length restricted Schur function formula may be also be identified with characters in the same way.

In each case the column length restriction owes its origin to the bijective correspondence between irreps of the dual groups specified by $p(\lambda)$ and $\lambda$. 
Dual pairs in spin modules

The spin modules $\Delta$ of $O(N)$ give rise to the following dual pairs of subgroups $G \times H$:

- $O(4np) \supseteq SO(2n) \times O(2p)$
- $O(4np + 2p) \supseteq SO(2n + 1) \times O(2p)$
- $O(4np + 2n) \supseteq SO(2n) \times O(2p + 1)$
- $O(4np + 2n + 2p + 1) \supseteq SO(2n + 1) \times O(2p + 1)$
- $O(4np) \supseteq Sp(2n) \times Sp(2p)$
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- $O(4np + 2n + 2p + 1) \supseteq SO(2n + 1) \times O(2p + 1)$
- $O(4np) \supseteq Sp(2n) \times Sp(2p)$

The dual pairs may be found by
- verifying that the actions of $G$ and $H$ mutually centralize one another
- determining multiplicity free common highest weight vectors of $G$ and $H$ [Hasegawa 89]
Dual pairs in spin modules

Each dual pair gives rise to an identity of characters of the form
\[ \text{ch } V_{O(N)}^\Delta = \sum_{k \in K} \text{ch } V_G^{\lambda(k)} \text{ch } V_H^{\mu(k)} \]

Such identities have been derived by

- Using the Laplace expansion of \( \text{ch } V_{O(N)}^\Delta \)
  - orthogonal subgroup case [Morris 58, 61]
  - symplectic subgroup case [Bump and Gamburd 05]
- Using a Robinson-Schensted-Knuth-Berele procedure in the symplectic subgroup case [Terada 91]
Dual pairs in spin modules

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\[ \text{ch } V_{O(N)}^\Delta = \sum_{k \in K} \text{ch } V_G^\lambda(k) \text{ch } V_H^\mu(k) \]

Such identities have been derived by

- Using the Laplace expansion of \( \text{ch } V_{O(N)}^\Delta \)
  - orthogonal subgroup case [Morris 58, 61]
  - symplectic subgroup case [Bump and Gamburd 05]
- Using a Robinson-Schensted-Knuth-Berele procedure in the symplectic subgroup case [Terada 91]

Here, in the symplectic subgroup case, we offer an alternative derivation based on a jeu-de-taquin procedure.
Semistandard Young tableaux

Let $\mathcal{T}^\lambda(n)$ be the set of $gl(n)$-tableaux $T$ obtained by filling the boxes of $F^\lambda$ with entries from $\{1 < 2 < \ldots < n\}$ such that they

T1 weakly increase across each row from left to right;
T2 strictly increase down each column from top to bottom;
Semistandard Young tableaux

Let $T^\lambda(n)$ be the set of $gl(n)$-tableaux $T$ obtained by filling the boxes of $F^\lambda$ with entries from $\{1 < 2 < \ldots < n\}$ such that they

T1 weakly increase across each row from left to right;

T2 strictly increase down each column from top to bottom;

Ex: For $n = 6$, $\lambda = (3, 3, 2)$ we have

$$T = \begin{array}{ccc}
1 & 2 & 3 \\
3 & 4 & 4 \\
4 & 5 \\
\end{array} \in T^{332}(6)$$
For $x = (x_1, x_2, \ldots, x_n)$ and any $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ let $x^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_n^{\kappa_n}$

Then

$$\text{ch} \ V^\lambda_{GL(n)} = s_\lambda(x) = \sum_{T \in T^\lambda(n)} x^{\text{wgt}(T)}$$

where $\text{wgt}(T)_k = \# k \in T$ for $k = 1, 2, \ldots, n$
Schur functions and tableaux

- For $x = (x_1, x_2, \ldots, x_n)$ and any $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ let $x^\kappa = x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_n^{\kappa_n}$

- Then

$$
\text{ch } V^\lambda_{GL(n)} = s_\lambda(x) = \sum_{T \in T^\lambda(n)} x^\text{wgt}(T)
$$

where $\text{wgt}(T)_k = \# k \in T$ for $k = 1, 2, \ldots, n$

- Ex: For $n = 6$, $\lambda = (3, 3, 2)$

$$
T = \begin{array}{ccc}
1 & 2 & 3 \\
3 & 4 & 4 \\
4 & 5 & \\
\end{array}
$$

$x^\text{wgt}(T) = x_1 x_2 x_3^2 x_4^3 x_5$
Let $SpT^\lambda(n)$ be the set of $sp(2n)$-tableaux $T$ obtained by filling the boxes of $F^\lambda$ with entries from $\{1 \prec 2 \prec \cdots \prec n\}$ such that they

S1 weakly increase across each row from left to right;

S2 strictly increase down each column from top to bottom;

S3 $k$ and $\bar{k}$ appear no lower than the $k$th row.
Let $SpT^\lambda(n)$ be the set of $sp(2n)$-tableaux $T$ obtained by filling the boxes of $F^\lambda$ with entries from $\{1 < 1 < 2 < 2 < \cdots < n < n\}$ such that they

- S1 weakly increase across each row from left to right;
- S2 strictly increase down each column from top to bottom;
- S3 $k$ and $\overline{k}$ appear no lower than the $k$th row.

Ex: For $n = 4$, $\lambda = (3, 3, 2, 1)$

\[
T = \begin{array}{ccc}
\overline{1} & 2 & 3 \\
2 & 3 & 3 \\
\overline{3} & 4 \\
4 \\
\end{array} \quad \in \quad SpT^{3321}(4)
\]
Symplectic characters and tableaux

Let $x = (x_1, x_2, \ldots, x_n)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ with $\bar{x}_k = x_k^{-1}$ for $k = 1, 2, \ldots, n$

Then

$$\text{ch } V^\lambda_{Sp(2n)} = sp_\lambda(x, \bar{x}) = \sum_{T \in SpT^\lambda(n)} x^{\text{wgt}(T)}$$

where $\text{wgt}(T)_k = \# k \in T - \# \bar{k} \in T$ for $k = 1, 2, \ldots, n$
Let $x = (x_1, x_2, \ldots, x_n)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ with $\bar{x}_k = x_k^{-1}$ for $k = 1, 2, \ldots, n$.

Then

$$\text{ch} V_{Sp(2n)}^\lambda = sp_\lambda (x, \bar{x}) = \sum_{T \in S_p T^\lambda (n)} x^{\text{wgt}(T)}$$

where $\text{wgt}(T)_k = \# k \in T - \# \bar{k} \in T$ for $k = 1, 2, \ldots, n$.

Ex: For $n = 4$, $\lambda = (3, 3, 2, 1)$

$$T = \begin{array}{ccc}
\bar{1} & \bar{2} & \bar{3} \\
2 & 3 & 3 \\
\bar{3} & 4 \\
4 
\end{array}$$

$$\text{wgt} (T) = x_1^{0-1} x_2^{1-1} x_3^{2-2} x_4^{2-0} = x_1^{-1} x_4^2$$
Dual pair identity

The identity to be proved takes the form

\[
\text{ch } V_{O(4np)}^\Delta = \sum_{\lambda \subseteq n^p} \text{ch } V_{Sp(2n)}^{\lambda\dagger} \text{ch } V_{Sp(2p)}^\lambda = \sum_{\lambda \subseteq p^n} \text{ch } V_{Sp(2n)}^\lambda \text{ch } V_{Sp(2p)}^{\lambda\dagger}
\]
The identity to be proved takes the form

\[ \text{ch } V^\Delta_{O(4np)} = \sum_{\lambda \subseteq n^p} \text{ch } V^\lambda_{Sp(2n)} \text{ch } V^\lambda_{Sp(2p)} = \sum_{\lambda \subseteq p^n} \text{ch } V^\lambda_{Sp(2n)} \text{ch } V^\lambda_{Sp(2p)} \]

where

\[ \text{ch } V^\lambda_{Sp(2n)} \text{ch } V^\lambda_{Sp(2p)} = sp \lambda(x, \overline{x}) \text{ sp } \lambda^\dagger(y, \overline{y}) \]

and

\[
\text{ch } V^\Delta_{O(4np)}(xy, x\overline{y}, \overline{x}y, \overline{x}\overline{y}) = \prod_{i=1}^{n} \prod_{j=1}^{p} (x_i^{\frac{1}{2}} y_j^{\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}})(x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}}) = \prod_{i=1}^{n} \prod_{j=1}^{p} (x_i + \overline{x}_i + y_j + \overline{y}_j) \]
Pairs of symplectic tableau

Let $\mathcal{R}(n, p)$ be the set of tableaux $R = (TS^\dagger)$ composed, for some $\lambda \subseteq (p^n)$, of $T \in S_{pT^\lambda}(n)$ and $S \in S_{pT^\lambda}(p)$ reoriented so as to constitute a rectangular tableaux of shape $F(p^n)$.

**Ex:** $n = 4, \ p = 5, \ \lambda = (3, 3, 2, 1), \ \lambda^\dagger = (4, 4, 2, 1, 0)$

\[
T = \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 3 \\
3 & 4 \\
4 \\
\end{array}
\quad
S = \begin{array}{cccc}
1' & 1' & 1' & 2' \\
2' & 4' & 4' & 4' \\
4' & 4' \\
5' \\
\end{array}
\quad
R = \begin{array}{ccccc}
1 & 2 & 3 & 4' & 2' \\
2 & 3 & 3 & 4' & 1' \\
3 & 4 & 4' & 4' & 1' \\
4 & 5' & 4' & 2' & 1' \\
\end{array}
\]
Observation

\[
\sum_{\lambda \subseteq p^n} sp_{\lambda}(x, \bar{x}) \ sp_{\lambda^\dagger}(y, \bar{y}) \\
= \sum_{\lambda \subseteq p^n} \sum_{T \in SpT^\lambda(n)} x^{wgt(T)} \sum_{S \in SpT^\lambda(n)} y^{wgt(S)} \\
= \sum_{R \in \mathcal{R}(n,p)} (x \ y)^{wgt(R)}
\]
Observation

\[
\sum_{\lambda \subseteq p^n} sp_\lambda(x, \overline{x}) \ sp_{\lambda^\dagger}(y, \overline{y})
= \sum_{\lambda \subseteq p^n} \sum_{T \in SpT^\lambda(n)} x^{wgt(T)} \sum_{S \in SpT^\lambda(n)} y^{wgt(S)}
= \sum_{R \in R(n,p)} (x \ y)^{wgt(R)}
\]

\[\textbf{Ex: } n = 4, \ p = 5, \ \lambda = (3, 3, 2, 1), \ \lambda^\dagger = (4, 4, 2, 1, 0)\]

\[
R = \begin{array}{cccccc}
1 & 2 & 3 & 4' & 2' \\
2 & 3 & 3 & 4' & 1' \\
3 & 4 & 4' & 4' & 1' \\
4 & 5' & 4' & 2' & 1'
\end{array}
\]

\[(x \ y)^{wgt(R)} = x_1^{-1} x_4^2 y_1 y_4^{-1} y_5\]
New rectangular tableaux

Let \( D(n, p) \) be the set of tableaux \( D \) obtained by filling the boxes of \( F(p^n) \) with entries from
\[
\{1 < 2 < \cdots < n < 1' < 2' < \cdots < p' < p'\}
\]
in such a way that:

D1 each unprimed entry \( k \) or \( \bar{k} \) lies in the \( k \)th row counted from top to bottom;

D2 each primed entry \( k' \) or \( \bar{k}' \) lies in the \( k \)th column counted from right to left.
New rectangular tableaux

Let $\mathcal{D}(n, p)$ be the set of tableaux $D$ obtained by filling the boxes of $F(p^n)$ with entries from

$$\{1 < 1 < 2 < \cdots < n < 1' < 1' < 2' < \cdots < p' < p'\}$$

in such a way that:

D1 each unprimed entry $k$ or $k'$ lies in the $k$th row counted from top to bottom;

D2 each primed entry $k'$ or $k'$ lies in the $k$th column counted from right to left.

Typically $D = \begin{array}{cccc}
1 & 1 & 2' & 1' \\
5' & 4' & 2 & 2 \\
3 & 4' & 3 & 2' \\
4 & 4' & 4 & 1' \\
\end{array}$ \quad \in \mathcal{D}(4, 5)
Metaplectic character

\[ \prod_{i=1}^{n} \prod_{j=1}^{p} (x_i + \overline{x}_i + y_j + \overline{y}_j) = \sum_{D \in D(n,p)} (x y)^{wgt(D)} \]

- \((x, y) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p)\)

- \(wgt(D)_i = \#k - \#\overline{k} \text{ for } i = k \text{ with } k = 1, 2, \ldots, n\)

- \(wgt(D)_i = \#k' - \#\overline{k}' \text{ for } i = n + k \text{ with } k = 1, 2, \ldots, p\)
Metaplectic character

\[\prod_{i=1}^{n} \prod_{j=1}^{p} (x_i + \bar{x}_i + y_j + \bar{y}_j) = \sum_{D \in D(n,p)} (x \ y)^{\text{wgt}(D)}\]

- \((x, y) = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_p)\)

- \(\text{wgt}(D)_i = \# k - \# \bar{k}\) for \(i = k\) with \(k = 1, 2, \ldots, n\)

- \(\text{wgt}(D)_i = \# k' - \# \bar{k}'\) for \(i = n + k\) with \(k = 1, 2, \ldots, p\)

**Ex:** \(D = \begin{array}{cccccc}
1' & 1 & 1 & 2' & 1' & -1 \\
5' & 4' & 2 & 2' & 2 & 0 \\
3 & 4' & 3 & 2' & 1' & 0 \\
4 & 4' & 4 & 2' & 1' & 2 \\
1 & -1 & 0 & 0 & 1 & \end{array}\)

\[\Rightarrow (x, y)^{\text{wgt}(D)} = x_1^{-1} x_4^2 y_1 y_4^{-1} y_5\]

**Note:** Entry in the \((i, j)\)th box is any one of \(\{i, \bar{i}, j, \bar{j}\}\)
Lemma

For all \( n, p \in \mathbb{N} \)

\[
\sum_{R \in \mathcal{R}(n,p)} (x \ y)^{\text{wgt}(R)} = \sum_{D \in \mathcal{D}(n,p)} (x \ y)^{\text{wgt}(D)}
\]
Lemma

For all \( n, p \in \mathbb{N} \)

\[
\sum_{R \in \mathcal{R}(n,p)} (x \ y)^{\text{wgt}(R)} = \sum_{D \in \mathcal{D}(n,p)} (x \ y)^{\text{wgt}(D)}
\]

- Construct a weight preserving bijection between \( \mathcal{R}(n, p) \) and \( \mathcal{D}(n, p) \).

- Use jeu de taquin to map each \( R \in \mathcal{R}(n, p) \) to corresponding \( D \in \mathcal{D}(n, p) \).

- Move each primed entry \( k' \) or \( \overline{k}' \) north-west to its own column, the \( k \)th, and then north while moving each unprimed entry \( i \) or \( \overline{i} \) to its own row, the \( i \)th.

- To right of \( k \)th column maintain \( S^{1-S^{3}} \) and \( S^{1^{\dagger}-S^{3^{\dagger}}} \).
Legitimate moves for $k'$

$k'$ in position $(i, j)$ with $i > 1$ and $j < k$

\[
\begin{align*}
&\begin{cases}
  k' & \text{if } a \leq b \\
  a & \text{if } a > b
\end{cases}
\end{align*}
\]
Legitimate moves for $k'$

- $k'$ in position $(i, j)$ with $i > 1$ and $j < k$

\[
\begin{cases} 
  a & b \\
  k' & a \\
\end{cases} \quad \iff 
\begin{cases} 
  k' & a \\
  b & a \\
\end{cases} \quad \text{if } a \leq b
\]

\[
\begin{cases} 
  a & b \\
  k' & a \\
\end{cases} \quad \iff 
\begin{cases} 
  k' & a \\
  k' & a \\
\end{cases} \quad \text{if } a > b
\]

- $k'$ in position $(1, j)$ with $j < k$

\[
\begin{cases} 
  a & k' \\
\end{cases} \quad \iff 
\begin{cases} 
  k' & a \\
\end{cases}
\]
Legitimate moves for $k'$

- $k'$ in position $(i, k)$ with $i > 1$

\[
\begin{array}{c|c|c}
| b & \hline \\
| \hline \\
| k' & \hline \\
| k' & \hline \\
\end{array}
\]

$\iff$

\[
\begin{array}{c|c|c}
| k' & \hline \\
| b & \hline \\
\end{array}
\]

if $b \leq i$
Legitimate moves for $k'$

- $k'$ in position $(i, k)$ with $i > 1$

\[
\begin{array}{c|c}
\hline
b & k' \\
\hline
k' & b \\
\hline
\end{array}
\iff b \leq i

Note

- $k'$ allowed by S1$^\dagger$
- $k'$ forbidden by S2$^\dagger$
Legitimate moves for $\overline{k}'$

In position $(i, j)$ with $i > 1$ and $j < k$

\[
\begin{array}{c|c}
\text{a} & \overline{k}' \\
\hline
\overline{k}' & \text{b} \\
\end{array}
\quad \iff 
\begin{array}{c|c}
\text{b} & \overline{k}' \\
\hline
\text{a} & \overline{k}' \\
\end{array}
\quad \begin{cases}
\text{if } a \leq b \\
\text{if } a > b
\end{cases}
\]
Legitimate moves for $\overline{k}'$

- $\overline{k}'$ in position $(i, j)$ with $i > 1$ and $j < k$

\[
\begin{align*}
\text{if } a &\leq b \\
\overline{k}' &\quad a \quad b \\
\text{if } a &> b \\
\overline{k}' &\quad b \\
&\quad \overline{k}' \quad a
\end{align*}
\]

- $\overline{k}'$ in position $(1, j)$ with $j < k$

\[
\begin{align*}
\overline{k}' &\quad a \\
\overline{k}' &\quad a
\end{align*}
\]
Legitimate moves for $\overline{k}'$

- $\overline{k}'$ in position $(i, k)$ with $i > 1$

\[
\begin{array}{c|c|c|c}
& b & \overline{k}' & \overline{k} \\
\hline
\overline{k}' & \overline{k}' & & \overline{k} \\
\hline
b & & & b
\end{array}
\]

if $b \leq i$
Legitimate moves for $\overline{k'}$

- $\overline{k'}$ in position $(i, k)$ with $i > 1$

\[
\begin{array}{c|c}
 b & \overline{k'} \\
\hline
 \overline{k'} & b \\
\end{array} \iff
\begin{array}{c|c}
 \overline{k'} & \overline{k'} \\
\hline
 \overline{k'} & b \\
\end{array}
\text{ if } b \leq i
\]

- Note

\[
\begin{array}{c|c}
 \overline{k'} & \overline{k'} \\
\hline
 \overline{k'} & \overline{k'} \\
\end{array}
\text{ allowed by } S1^\dagger
\]
\[
\begin{array}{c|c}
 \overline{k'} & \overline{k'} \\
\hline
 \overline{k'} & \overline{k'} \\
\end{array}
\text{ forbidden by } S2^\dagger
\]
No transformations necessary

Note

\[ \begin{array}{c}
\bar{\alpha} \\
\bar{\bar{\alpha}} \\
\end{array} \quad \text{and} \quad \begin{array}{c}
\bar{\beta} \\
\bar{\bar{\beta}} \\
\end{array} \]

allowed by S1
No transformations necessary

- Note

\[ i \, i \] and \[ i \, i \] allowed by S1

- Note

\[ i \, i \] and \[ i \, i \] forbidden by S2
Weight preserving transformations

\( k' \) in position \( (i, k) \) so that \( k' \) is in \( k \)th column, but blocks \( \overline{k'} \) from moving to \( k \)th column

\[
\begin{array}{c|c}
  k' & \overline{k'} \\
\end{array} \iff \begin{array}{c|c}
  i & \overline{i}
\end{array}
\]
Weight preserving transformations

- \( k' \) in position \((i, k)\) so that \( k' \) is in \( k \)th column, but blocks \( \overline{k'} \) from moving to \( k \)th column

\[
\begin{array}{c}
| k' | \overline{k'} \\
\end{array} \leftrightarrow \begin{array}{c}
| i | \overline{i} \\
\end{array}
\]

- \( i \) in position \((i, k)\) so that \( i \) is in \( i \)th row, but blocks \( \overline{i} \) from moving to \( i \)th row

\[
\begin{array}{c}
\overline{i} \\
| i |
\end{array} \leftrightarrow \begin{array}{c}
| \overline{k'} | \\
| k' |
\end{array}
\]
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$:th column and then move N as far as possible in this column.
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify large primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

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Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4' & 2' \\
2 & 3 & 3 & 4' & 1' \\
3 & 4 & 4' & 4' & 1' \\
4 & 5' & 4' & 2' & 1' \\
\end{array}
\]
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

\[
\begin{array}{cccccc}
\overline{1} & 2 & \overline{3} & 4' & 2' \\
2 & 3 & 3 & \overline{4'} & 1' \\
\overline{3} & 5' & 4' & \overline{4'} & 1' \\
4 & 4 & \overline{4'} & 2' & \overline{1'}
\end{array}
\]
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$-th column and then move N as far as possible in this column
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column

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Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$\textsuperscript{th} column and then move N as far as possible in this column.

```
   1   4'  2   3   2'
   5'  4'  2   4'  1'
   3   3   3   4'  1'
   4   4   4'  2'  1'
```
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column

```
 1 2' 3 2' 
 5' 4' 2 4' 1' 
 3 3 3 4' 1' 
 4 4 4' 2' 1' 
```
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

\[
\begin{array}{ccccc}
\bar{1} & 4' & \bar{2} & 4' & 2' \\
5' & 4' & 2 & 3 & 1' \\
\bar{3} & 3 & 3 & 4' & 1' \\
4 & 4 & 4' & 2' & 1'
\end{array}
\]
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

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Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

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Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

 Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

\[
\begin{array}{cccccc}
\bar{1} & 1 & \bar{1} & 2 & 2' \\
5' & 4' & 2 & \bar{3} & 1' \\
\bar{3} & \bar{4}' & 3 & 3 & 1' \\
4 & \bar{4}' & 4 & 2' & \bar{1}'
\end{array}
\]
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

```
   1   1   1   2   2'
5'  4'  2   3   1'
3   4'  3   3   1'
4   4'  4   2'  1'
```
Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.
Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column
Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$-th column and then move N as far as possible in this column.

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Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$-th column and then move N as far as possible in this column.

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Map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$

- Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$th column and then move N as far as possible in this column.

\[
\begin{array}{cccccc}
\bar{1} & 1 & \bar{1} & 2' & \bar{2} \\
5' & 4' & 2 & \bar{2}' & 1' \\
\bar{3} & \bar{4}' & 3 & 2' & 1' \\
4 & \bar{4}' & 4 & \bar{2}' & \bar{1}' \\
\end{array}
\]
Map from $\mathcal{R} \in \mathcal{R}(n, p)$ to $\mathcal{D} \in \mathcal{D}(n, p)$

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches $k$-th column and then move N as far as possible in this column.
Map from \( R \in \mathcal{R}(n, p) \) to \( D \in \mathcal{D}(n, p) \)

Identify largest primed entries. Move topmost such entry NW by a sequence of interchanges with nearest neighbours until it reaches \( k \)th column and then move N as far as possible in this column.

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Thus we have a map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$ illustrated by:

\[ R = \begin{array}{cccc}
1 & 2 & 3 & 4' \\
2 & 3 & 3 & 4' \\
3 & 4 & 4' & 4' \\
4 & 5' & 4' & 2' \\
\end{array} \quad \Leftrightarrow \quad \begin{array}{cccc}
1 & 1 & 1 & 2' \\
5' & 4' & 2 & 2' \\
3 & 4' & 3 & 2' \\
4 & 4' & 4 & 2' \\
\end{array} = D \]
Thus we have a map from $R \in \mathcal{R}(n, p)$ to $D \in \mathcal{D}(n, p)$ illustrated by:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4' \\
2 & 3 & 3 & 4' \\
3 & 4 & 4' & 1' \\
4 & 5' & 4' & 2' \\
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{cccc}
\bar{1} & 1 & \bar{1} & 2' & 1' \\
5' & 4' & 2 & \bar{2}' & \bar{2} \\
3 & \bar{4}' & 3 & 2' & 1' \\
4 & \bar{4}' & 4 & \bar{2}' & \bar{1}' \\
\end{array}
\]

- Every step is reversible - the map is bijective
- The map is weight preserving
- Hence our dual pair character identity is proven
Skew Young diagrams

- Given partitions $\lambda$ and $\mu$ such that all boxes of $F^{\mu}$ are contained in $F^{\lambda}$ we write $\mu \subseteq \lambda$.

- Removing the boxes of $F^{\mu}$ from $F^{\lambda}$ leaves the skew Young diagram $F^{\lambda/\mu}$.

- Ex: $\lambda = (5, 4, 2)$, $\mu = (3, 1)$, $F^{\lambda/\mu} = \begin{array}{ccc} \ast & \ast & \ast \\ \ast & & \end{array}$
Skew Young diagrams

- Given partitions $\lambda$ and $\mu$ such that all boxes of $F^\mu$ are contained in $F^\lambda$ we write $\mu \subseteq \lambda$.

- Removing the boxes of $F^\mu$ from $F^\lambda$ leaves the skew Young diagram $F^{\lambda/\mu}$

- Ex: $\lambda = (5, 4, 2), \mu = (3, 1), F^{\lambda/\mu} = \begin{array}{|c|c|c|}
\hline
* & * & * \\
\hline
\end{array}$

- Let $T^{\lambda/\mu}(n)$ be the set of $gl(n)$-tableaux $T$ obtained by filling the boxes of $F^{\lambda/\mu}$ with entries from $\{1 < 2 < \ldots < n\}$ such that they
  
  T1 weakly increase across each row from left to right
  
  T2 strictly increase down each column from top to bottom
For \( x = (x_1, x_2, \ldots, x_n) \) with \( n \in \mathbb{N} \)

\[
s_{\lambda/\mu}(x) = \sum_{T \in \mathcal{T}^{\lambda/\mu}(n)} x^{\text{wgt}(T)}
\]

where \( \text{wgt}(T)_k = \# k \in T \) for \( k = 1, 2, \ldots, n \)
Skew Schur function

For $x = (x_1, x_2, \ldots, x_n)$ with $n \in \mathbb{N}$

$$s_{\lambda/\mu}(x) = \sum_{T \in T^{\lambda/\mu}(n)} x^{\text{wgt}(T)}$$

where $\text{wgt}(T)_k = \# k \in T$ for $k = 1, 2, \ldots, n$

Ex: $n = 6$, $\lambda = (5, 4, 2)$, $\mu = (3, 1)$

$$T^{\lambda/\mu} = \begin{array}{cccc}
* & * & * & 2 & 3 \\
* & 1 & 4 & 4 \\
1 & 5 & & \\
\end{array}$$

$$x^{\text{wgt}(T)} = x_1^2 x_2 x_3 x_4^2 x_5$$
Schur function expansion

For \( x = (x_1, x_2, \ldots, x_m), \ y = (y_1, y_2, \ldots, y_n) \) with \( m, n \in \mathbb{N} \)

\[
s_\lambda(x, y) = \sum_\mu s_\mu(x) \ s_{\lambda/\mu}(y)
\]
Schur function expansion

For $x = (x_1, x_2, \ldots, x_m)$, $y = (y_1, y_2, \ldots, y_n)$ with $m, n \in \mathbb{N}$,

$$s_\lambda(x, y) = \sum_\mu s_\mu(x) s_{\lambda/\mu}(y)$$

Ex: $m = 4$, $n = 6$, $\lambda = (5, 4, 2)$, $\mu = (3, 1)$

\[
\begin{array}{cccc}
1 & 3 & 3 & 2 \\
4 & 1 & 4 & 4 \\
1 & 5 \\
\end{array}
\]

$$(x \ y)^{\text{wgt} (T)} = x_1 x_3^2 x_4 y_1^2 y_2 y_3 y_4^2 y_5$$
Cauchy formula and its inverse

Let $m, n$ be positive integers.

Then for all $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$

$$\sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)^{-1}$$

$$\sum_{\lambda} (-1)^{|\lambda|} s_{\lambda}(x) s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)$$
Cauchy formula and its inverse

Let $m, n$ be positive integers

Then for all $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$

$$
\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j}
$$

$$
\sum_{\lambda} (-1)^{|\lambda|} s_\lambda(x) s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)
$$

The first sum over $\lambda$ is infinite with non-zero terms arising for all $\ell(\lambda) \leq \min\{m, n\}$, and no restriction on $\ell(\lambda')$

The second sum over $\lambda$ is finite, with $\lambda \subseteq n^m$, since $s_\lambda(x) = 0$ if $\ell(\lambda) > m$ and $s_{\lambda'}(y) = 0$ if $\ell(\lambda') > n$
Determinantal identity

For \( x = (x_1, x_2, \ldots, x_m) \), \( y = (y_1, y_2, \ldots, y_n) \) with \( m, n \in \mathbb{N} \)

\[
\sum_{\lambda} (-1)^{|\lambda|} s_{\lambda}(x) s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)
\]

\[
= \frac{1}{ \begin{vmatrix} x_i^{m-j} & y_{a}^{n-b} \\ \vdots & \vdots \\ x_i^{m+n-j} & x_i^{m+n-j} \end{vmatrix} } \cdot \begin{vmatrix} y_{n+1-i}^{j-1} \\ \vdots \\ y_{n+1-i}^{j-1} \end{vmatrix}
\]

The \((m+n) \times (m+n)\) determinant is partitioned after the \(n\)th row
Determinantal identity

For \( x = (x_1, x_2, \ldots, x_m) \), \( y = (y_1, y_2, \ldots, y_n) \) with \( m, n \in \mathbb{N} \)

\[
\sum_{\lambda} (-1)^{|\lambda|} \ s_{\lambda}(x) \ s_{\lambda'}(y) = \prod_{i=1}^{m} \prod_{j=1}^{n} (1 - x_i y_j)
\]

\[
= \frac{1}{\begin{vmatrix} x_i^{m-j} & y_{n+1-i}^n \end{vmatrix} \begin{vmatrix} y_i^{n-b} \end{vmatrix} \begin{vmatrix} x_i^{m+n-j} \end{vmatrix}} \cdot \begin{vmatrix} y_n^{j-1} \end{vmatrix} \begin{vmatrix} \cdots \end{vmatrix} \begin{vmatrix} x_i^{m+n-j} \end{vmatrix}
\]

The \((m+n) \times (m+n)\) determinant is partitioned after the \(n\)th row

Proof Use either Laplace expansion to obtain Schur functions directly, or three Vandermonde identities to obtain product form
Theorem [Kwon 08, Hamel and K. 08]

Let \( x = (x_1, \ldots, x_m) \), \( y = (y_1, \ldots, y_n) \) with \( m, n \geq 1 \).

Then for all \( p \geq 0 \) we have

\[
\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) s_\lambda(y) = (y_1 y_2 \cdots y_n)^p \ s_p^n(x, y)
\]

\[
= \frac{1}{|x_i^{m-j}| \ |y_a^{n-b}| \ \prod_{i=1}^m \prod_{a=1}^n (1 - x_i y_a)} \cdot \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|} \ s_\sigma(x) \ s_\tau(y)
\]

where \( \sigma = (\zeta + p^r) \) and \( \tau = (\zeta' + p^r) \) with \( r = r(\zeta) \).
Proof

For \( x = (x_1, x_2, \ldots, x_m) \), \( y = (y_1, y_2, \ldots, y_n) \) and \( p \in \mathbb{N} \)

\[
s_{p^n}(x, \overline{y}) = \sum_{\zeta \subseteq p^n} s_{\zeta}(x) s_{p^n/\zeta}(y) = \sum_{T \in \mathcal{T} p^n (m+n)} (x \overline{y})^{\text{wgt} (T)}
\]
Proof

For \( x = (x_1, x_2, \ldots, x_m) \), \( y = (y_1, y_2, \ldots, y_n) \) and \( p \in \mathbb{N} \)

\[
sp^n(x, \overline{y}) = \sum_{\zeta \subseteq p^n} s_{\zeta}(x) \quad sp^n/\zeta(y) = \sum_{T \in \mathcal{T}^n(m+n)} (x \overline{y})^{\text{wgt}(T)}
\]

Ex: Typically, for \( m = 6, n = 4, p = 5 \), and order \( 1 < 2 < 3 < 4 < 5 < 6 < \overline{4} < \overline{3} < \overline{2} < \overline{1} \) we have

\[
T = \begin{bmatrix}
1 & 2 & 3 & 5 & \overline{4} \\
2 & 4 & 4 & \overline{4} & \overline{3} \\
5 & \overline{4} & \overline{4} & 2 & 2 \\
\overline{3} & \overline{2} & \overline{1} & \overline{1} & \overline{1}
\end{bmatrix}
\]
Proof

However, separating the blue entries from the red entries and taking the complement of each column of the latter with respect to \(1 \ 2 \ 3 \ 4\) gives

\[
\begin{array}{cccc}
1 & 2 & 3 & 5 \\
2 & 4 & 4 & 4 \\
5 & 4 & 4 & 2 \\
3 & 2 & 1 & 1 \\
\end{array}
\quad\Leftrightarrow\quad
\begin{array}{cccc}
1 & 2 & 3 & 5 \\
2 & 4 & 4 & \\
5 & & & \\
3 & 2 & 1 & 1 \\
\end{array}
\cdot
\begin{array}{cccccccc}
 \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} & \overline{1} \\
 \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} & \overline{2} \\
 \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} & \overline{3} \\
 \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} & \overline{4} \\
\end{array}
\cdot
\begin{array}{cccc}
1 & 1 & 2 & 3 \\
2 & 3 & 3 & \\
2 & 3 & 3 & \\
4 & & & \\
\end{array}
\]
Proof

- However, separating the blue entries from the red entries and taking the complement of each column of the latter with respect to $1 \ 2 \ 3 \ 4$ gives

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Hence, setting $y = y_1 y_2 \cdots y_n$ we have

$$s_{p^n}(x, \overline{y}) = y^{-p} \sum_{\zeta \subseteq p^n} s_{\zeta}(x) \ s_{\zeta}(y)$$
Proof

It follows that

$$\sum_{\zeta : \ell(\zeta') \leq p} s_{\zeta}(x) \ s_{\zeta}(y) = y^p \ s_{p^n}(x, y)$$

$$= \frac{1}{|x^{m-j}_i| \ |y^{n-b}_a| \ \prod_{i=1}^{m} \prod_{a=1}^{n} (1 - x_iy_a)} \cdot$$
Lemma [K. 2008]

Let \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_n) \)

Then for each pair of integers \( p \) and \( q \) we have

\[
\frac{1}{\begin{vmatrix} x_i^{m-j} & y_i^{n-j} \\ \end{vmatrix}} = \sum_{\zeta \subseteq n^m} (-1)^{|\zeta|} s_{\zeta+p^r(\zeta)}(x) s_{\zeta'+p^r(\zeta)}(y)
\]

where the large determinant is \((m + n) \times (m + n)\), and is partitioned after the \( n \)th row and \( n \)th column.
Lemma contd.

If \( \zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m) \)

with \( a_1 < n, \ b_1 < m \) and \( r = r(\zeta) \), then

\( \zeta + p^r = \begin{pmatrix} a_1 + p & a_2 + p & \cdots & a_r + p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \)

\( \zeta' + p^r = \begin{pmatrix} b_1 + q & b_2 + q & \cdots & b_r + q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \)

with \( a_r \geq \max\{0, -p\} \) and \( b_r \geq \max\{0, -q\} \)
Lemma contd.

If \( \zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m) \)

with \( a_1 < n, \ b_1 < m \) and \( r = r(\zeta) \), then

\[ \zeta + p^r = \begin{pmatrix} a_1 + p & a_2 + p & \cdots & a_r + p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \]

\[ \zeta' + p^r = \begin{pmatrix} b_1 + q & b_2 + q & \cdots & b_r + q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \]

with \( a_r \geq \max\{0, -p\} \) and \( b_r \geq \max\{0, -q\} \)

Proof: By Laplace expansion
Examples of key determinant

\[ m = 3, \quad n = 4, \quad p = 2, \quad q = 1 \]
Examples of key determinant

\[ m = 3, \ n = 4, \ p = 2, \ q = 1 \]

|    | \( y_1 \) | \( y_1^2 \) | \( y_1^3 \) | \( y_2 \) | \( y_2^2 \) | \( y_2^3 \) | \( y_3 \) | \( y_3^2 \) | \( y_3^3 \) | \( y_4 \) | \( y_4^2 \) | \( y_4^3 \) | \( y_5 \) | \( y_5^2 \) | \( y_5^3 \) | \( y_6 \) | \( y_6^2 \) | \( y_6^3 \) | \( y_7 \) | \( y_7^2 \) | \( y_7^3 \) |
|----|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 1  | 1          | \( y_4 \)  | \( y_4^2 \) | \( y_4^3 \) | \( y_4^5 \)  | \( y_4^6 \)  | \( y_4^7 \)  | \( y_3 \)  | \( y_3^2 \)  | \( y_3^3 \)  | \( y_3^5 \)  | \( y_3^6 \)  | \( y_3^7 \)  | \( y_2 \)  | \( y_2^2 \)  | \( y_2^3 \)  | \( y_2^5 \)  | \( y_2^6 \)  | \( y_2^7 \)  | \( y_1 \)  | \( y_1^2 \)  | \( y_1^3 \)  |
| 1  | \( y_3 \)  | \( y_3^2 \) | \( y_3^3 \) | \( y_3^5 \)  | \( y_3^6 \)  | \( y_3^7 \)  | \( y_2 \)  | \( y_2^2 \)  | \( y_2^3 \)  | \( y_2^5 \)  | \( y_2^6 \)  | \( y_2^7 \)  | \( y_1 \)  | \( y_1^2 \)  | \( y_1^3 \)  | \( y_1^5 \)  | \( y_1^6 \)  | \( y_1^7 \)  |
| 1  | \( y_2 \)  | \( y_2^2 \) | \( y_2^3 \) | \( y_2^5 \)  | \( y_2^6 \)  | \( y_2^7 \)  | \( y_1 \)  | \( y_1^2 \)  | \( y_1^3 \)  | \( y_1^5 \)  | \( y_1^6 \)  | \( y_1^7 \)  |
| 1  | \( y_1 \)  | \( y_1^2 \) | \( y_1^3 \) | \( y_1^5 \)  | \( y_1^6 \)  | \( y_1^7 \)  |
| ... | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        | ...        |
| \( x_1 \)^8 | \( x_1 \)^7 | \( x_1 \)^6 | \( x_1 \)^5 | \( x_1 \)^2 | \( x_1 \)  | 1          |
| \( x_2 \)^8 | \( x_2 \)^7 | \( x_2 \)^6 | \( x_2 \)^5 | \( x_2 \)^2 | \( x_2 \)  | 1          |
| \( x_3 \)^8 | \( x_3 \)^7 | \( x_3 \)^6 | \( x_3 \)^5 | \( x_3 \)^2 | \( x_3 \)  | 1          |
Examples of key determinant

- $m = 3$, $n = 4$, $p = -2$, $q = 1$
Examples of key determinant

\[ m = 3, \ n = 4, \ p = -2, \ q = 1 \]

\[
\begin{array}{cccccccc}
1 & y_4 & y_4^2 & y_4^3 & : & y_4^5 & y_4^6 & y_4^7 \\
1 & y_3 & y_3^2 & y_3^3 & : & y_3^5 & y_3^6 & y_3^7 \\
1 & y_2 & y_2^2 & y_2^3 & : & y_2^5 & y_2^6 & y_2^7 \\
1 & y_1 & y_1^2 & y_1^3 & : & y_1^5 & y_1^6 & y_1^7 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
x_1^4 & x_1^3 & - & - & : & x_1^2 & x_1 & 1 \\
x_2^4 & x_2^3 & - & - & : & x_2^2 & x_2 & 1 \\
x_3^4 & x_3^3 & - & - & : & x_3^2 & x_3 & 1
\end{array}
\]
Examples of key determinant

$m = 3$, $n = 4$, $p = 2$, $q = -1$
Examples of key determinant


text

\[ m = 3, \ n = 4, \ p = 2, \ q = -1 \]

\[
\begin{array}{cccc}
1 & y_4 & y_4^2 & y_4^3 & \vdots & - & y_4^4 & y_4^5 \\
1 & y_3 & y_3^2 & y_3^3 & \vdots & - & y_3^4 & y_3^5 \\
1 & y_2 & y_2^2 & y_2^3 & \vdots & - & y_2^4 & y_2^5 \\
1 & y_1 & y_1^2 & y_1^3 & \vdots & - & y_1^4 & y_1^5 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
x_1^8 & x_1^7 & x_1^6 & x_1^5 & \vdots & x_1^2 & x_1 & 1 \\
x_2^8 & x_2^7 & x_2^6 & x_2^5 & \vdots & x_2^2 & x_2 & 1 \\
x_3^8 & x_3^7 & x_3^6 & x_3^5 & \vdots & x_3^2 & x_3 & 1 \\
\end{array}
\]
Examples of key determinant

- $m = 3$, $n = 4$, $p = -2$, $q = -1$
Examples of key determinant

- $m = 3$, $n = 4$, $p = -2$, $q = -1$

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<td>$y_3^3$</td>
<td>$y_3^4$</td>
<td>$y_3^5$</td>
</tr>
<tr>
<td>1</td>
<td>$y_2$</td>
<td>$y_2^2$</td>
<td>$y_2^3$</td>
<td>$y_2^4$</td>
<td>$y_2^5$</td>
</tr>
<tr>
<td>1</td>
<td>$y_1$</td>
<td>$y_1^2$</td>
<td>$y_1^3$</td>
<td>$y_1^4$</td>
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<td>...</td>
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</tr>
<tr>
<td>$x_1^4$</td>
<td>$x_1^3$</td>
<td>$x_1^2$</td>
<td>$x_1$</td>
<td>1</td>
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</tr>
<tr>
<td>$x_2^4$</td>
<td>$x_2^3$</td>
<td>$x_2^2$</td>
<td>$x_2$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$x_3^4$</td>
<td>$x_3^3$</td>
<td>$x_3^2$</td>
<td>$x_3$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Ex 1: \( m = 3, \ n = 4, \ p = 2, \ q = 1 \)

Ex. \( \pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 5 & 7 & 1 & 4 & 6
\end{pmatrix} \)
Ex 1: \( m = 3, \ n = 4, \ p = 2, \ q = 1 \)

\[
\pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 3 & 5 & 7 & 1 & 4 & 6
\end{pmatrix}
\]

\[
\begin{array}{ccccccc}
1 & y_4 & y_4^2 & y_4^3 & : & y_4^5 & y_4^6 & y_4^7 \\ 1 & y_3 & y_3^2 & y_3^3 & : & y_3^5 & y_3^6 & y_3^7 \\ 1 & y_2 & y_2^2 & y_2^3 & : & y_2^5 & y_2^6 & y_2^7 \\ 1 & y_1 & y_1^2 & y_1^3 & : & y_1^5 & y_1^6 & y_1^7 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
x_1^8 & x_1^7 & x_1^6 & x_1^5 & : & x_1^2 & x_1 & 1 \\
x_2^8 & x_2^7 & x_2^6 & x_2^5 & : & x_2^2 & x_2 & 1 \\
x_3^8 & x_3^7 & x_3^6 & x_3^5 & : & x_3^2 & x_3 & 1
\end{array}
\]

\[
\sim - \begin{pmatrix}
y_4 & y_4^2 & y_4^5 & y_4^7 \\
y_3 & y_3^2 & y_3^5 & y_3^7 \\
y_2 & y_2^2 & y_2^5 & y_2^7 \\
y_1 & y_1^2 & y_1^5 & y_1^7
\end{pmatrix} \cdot \begin{pmatrix}
x_1 & x_5 & x_1 \\
x_2 & x_5 & x_2 \\
x_3 & x_5 & x_3
\end{pmatrix}
\]

\[
= - s_{4311}(y) \begin{vmatrix} y_{a-b} \end{vmatrix} \cdot s_{641}(x) \begin{vmatrix} x_{i-j}^3 \end{vmatrix}
\]
**Ex 1:** $m = 3$, $n = 4$, $p = 2$, $q = 1$

\[
\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 7 & 1 & 4 & 6 \end{pmatrix} \quad (-1)^\pi = (-1)^{|\zeta|} = -1
\]

\[
\zeta = (4, 2, 1) = \begin{pmatrix} 3 & 0 \\ 2 & 0 \end{pmatrix}
\]

\[
\sigma = \begin{pmatrix} 3+2 & 0+2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} = (6, 4, 1)
\]

\[
\zeta' = (3, 2, 1, 1) = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}
\]

\[
\tau = \begin{pmatrix} 2+1 & 0+1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix} = (4, 3, 1, 1)
\]
Ex 2: $m = 3$, $n = 4$, $p = -2$, $q = -1$

Ex. \( \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 6 & 7 & 1 & 2 & 5 \end{pmatrix} \)
Ex 2: \( m = 3, \ n = 4, \ p = -2, \ q = -1 \)

\[
\pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 4 & 6 & 7 & 1 & 2 & 5
\end{pmatrix}
\]

\[
\begin{array}{cccccc}
1 & y_4 & y_4^2 & y_4^3 & : & - y_4^4 & y_4^5 \\
1 & y_3 & y_3^2 & y_3^3 & : & - y_3^4 & y_3^5 \\
1 & y_2 & y_2^2 & y_2^3 & : & - y_2^4 & y_2^5 \\
1 & y_1 & y_1^2 & y_1^3 & : & - y_1^4 & y_1^5 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\sim - \begin{array}{cccccc}
y_4^2 & y_4^3 & y_4^4 & y_4^5 & : & x_1^4 & x_1^3 & x_1^2 \\
y_3^2 & y_3^3 & y_3^4 & y_3^5 & : & x_2^4 & x_2^3 & x_2^2 \\
y_2^2 & y_2^3 & y_2^4 & y_2^5 & : & x_3^4 & x_3^3 & x_3^2 \\
y_1^2 & y_1^3 & y_1^4 & y_1^5 & : & x_4^4 & x_4^3 & x_4^2
\end{array}
\cdot s_{2222}(y) \left| y_a^{4-b} \right| \cdot s_{222}(x) \left| x_i^{3-j} \right|
\]
**Ex 2:** $m = 3, \ n = 4, \ p = -2, \ q = -1$

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 4 & 6 & 7 & 1 & 2 & 5
\end{bmatrix}
\]

\[
(-1)^{\pi} = (-1)^{|\zeta'|} = +1
\]

\[
\zeta = (4, 4, 2) = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}
\]

\[
\begin{array}{c|c|c|c}
* & * & * & * \\
\hline
* & * & * & *
\end{array}
\]

\[
\sigma = \begin{pmatrix} 3 & -2 & 2 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = (2, 2, 2)
\]

\[
\zeta' = (3, 3, 2, 2) = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}
\]

\[
\begin{array}{c|c}
* \\
\hline
* \\
\end{array}
\]

\[
\tau = \begin{pmatrix} 2 & -1 & 1 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} = (2, 2, 2, 2)
\]
Proof of second part of Theorem

Just note that for $p \geq 0$

\[
\begin{align*}
&\begin{vmatrix}
  j-1 + x_{j>n,p} \\
  y_{n+1-i} \\
  \ldots \\
  m+n-j+x_{j\leq n,p} \\
  x_{i-n}
\end{vmatrix} = \\
&\begin{vmatrix}
  y_{n+1-i} \\
  \ldots \\
  x_{i-n}
\end{vmatrix}
\end{align*}
\]

Then use the Lemma with $p = q \geq 0$
Row length restricted Cauchy formula

For all $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and $p \geq 0$

$$
\sum_{\lambda: \ell(\lambda') \leq p} s_\lambda(x) s_\lambda(y)
= \frac{1}{\prod_{i=1}^{m} \prod_{a=1}^{n} (1 - x_i y_a)} \cdot \sum_{\zeta} (-1)^{|\zeta|} s_{\zeta+p^r}(x) s_{\zeta'+p^r}(y)
$$

$$
= \sum_{\lambda} s_\lambda(x) s_\lambda(y) \cdot \sum_{\zeta} (-1)^{|\zeta|} s_{\zeta+p^r}(x) s_{\zeta'+p^r}(y)
$$
Row length restricted Cauchy formula

For all \( x = (x_1, x_2, \ldots), \ y = (y_1, y_2, \ldots) \) and \( p \geq 0 \)

\[
\sum_{\lambda : \ell(\lambda') \leq p} s_{\lambda}(x) \ s_{\lambda}(y) = \frac{1}{\prod_{i=1}^{m} \prod_{a=1}^{n} (1 - x_i y_a)} \cdot \sum_{\zeta} (-1)^{|\zeta|} s_{\zeta + p^r}(x) \ s_{\zeta' + p^r}(y)
\]

\[
= \sum_{\lambda} s_{\lambda}(x) \ s_{\lambda}(y) \cdot \sum_{\zeta} (-1)^{|\zeta|} s_{\zeta + p^r}(x) \ s_{\zeta' + p^r}(y)
\]

This expresses the row length restricted series as a product of the unrestricted series times a correction factor.
Column length restricted Cauchy formula

Using the involutions $\omega_x : s_\lambda(x) \mapsto s_{\lambda'}(x)$ and $\omega_y : s_\lambda(y) \mapsto s_{\lambda'}(y)$ for all $\lambda$, either separately or together, we obtain three more restricted formula.
Column length restricted Cauchy formula

Using the involutions $\omega_x : s_\lambda(x) \mapsto s_{\lambda'}(x)$ and $\omega_y : s_\lambda(y) \mapsto s_{\lambda'}(y)$ for all $\lambda$, either separately or together, we obtain three more restricted formula.

Using $\omega_x \omega_y$ we find that for all $x$, $y$ and for all $p \geq 0$

$$\sum_{\lambda: \ell(\lambda) \leq p} s_\lambda(x) s_\lambda(y)$$

$$= \sum_\lambda s_\lambda(x) s_\lambda(y) \cdot \sum_\zeta (-1)^{|\zeta|} s_{(\zeta+p^r)'}(x) s_{(\zeta'+p^r)'}(y)$$

$$= \sum_\lambda s_\lambda(x) s_\lambda(y) \cdot \sum_\eta (-1)^{|\eta|} s_{\eta-p^r}(x) s_{\eta'-p^r}(y)$$

where the sum over $\eta$ is restricted to those $\eta$ such that both $\eta - p^r$ and $\eta' - p^r$ are partitions.
Column length restricted Cauchy formula

Using our Lemma with both $p$ and $q$ set equal to $-p$, with $p \geq 0$ gives

**Theorem** Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$ with $m, n \geq 1$. Then for all $p \geq 0$ we have

\[
\sum_{\lambda : \ell(\lambda) \leq p} s_\lambda(x) s_\lambda(y) = \frac{1}{|x_i^{m-j}| |y_a^{n-b}| \prod_{i=1}^{m} \prod_{a=1}^{n} (1 - x_i y_a)}
\]

\[
\begin{vmatrix}
  y_{j-1}^{n+1-i} & \ldots & \chi_{j \leq n-p} x_i^{m+n-j-p} \\
  y_{n+1-i}^{j-1} & \ldots & \chi_{j > n+p} y_n^{j-1-p} \\
  \chi_{j \leq n-p} x_i^{m+n-j-p} & \ldots & x_i^{m+n-j}
\end{vmatrix}
\]
Generalisation to the supersymmetric case

All our restricted row and column length formula involving symmetric functions may be generalised to the case of supersymmetric functions.
Generalisation to the supersymmetric case

- All our restricted row and column length formula involving symmetric functions may be generalised to the case of supersymmetric functions.
- Characters of Lie groups and algebras may be expressed in terms of symmetric Schur functions.
- Characters of Lie supergroups and superalgebras may be expressed in terms of supersymmetric Schur functions.
Supersymmetric functions

- Let $m, n$ be fixed positive integers
- Let $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$
- A function $f(x/y)$ is said to be supersymmetric if it is
  - symmetric under permutations of the $x_i$
  - symmetric under permutations of the $y_j$
  - independent of $t$ if $x_i = t = -y_j$ for any $i$ and $j$
Supersymmetric functions

- Let \( m, n \) be fixed positive integers
- Let \( x = (x_1, x_2, \ldots, x_m) \) and \( y = (y_1, y_2, \ldots, y_n) \)
- A function \( f(x/y) \) is said to be supersymmetric if it is
  - symmetric under permutations of the \( x_i \)
  - symmetric under permutations of the \( y_j \)
  - independent of \( t \) if \( x_i = t = -y_j \) for any \( i \) and \( j \)
- For each partition \( \lambda \) the supersymmetric Schur function \( s_\lambda(x/y) \) may be defined by
  \[
s_\lambda(x/y) = \sum_\mu s_\mu(x) s_{\lambda'/\mu'}(y)
  \]
Semistandard supertableaux

Let $\mathcal{T}^\lambda(m/n)$ be the set of $gl(m/n)$-tableaux $T$ obtained by filling the boxes of $F^\lambda$ with entries from $\{1 < 2 < \ldots < n < 1' < 2' < \ldots < n'\}$ such that unprimed entries

- $T_1$ weakly increase across each row from left to right
- $T_2$ strictly increase down each column from top to bottom

and primed entries

- $T'_1$ strictly increase across each row from left to right
- $T'_2$ weakly increase down each column from top to bottom
Semistandard supertableaux

Let $T^\lambda(m/n)$ be the set of $gl(m/n)$-tableaux $T$ obtained by filling the boxes of $F^\lambda$ with entries from $\{1 < 2 < \ldots < n < 1' < 2' < \ldots < n'\}$ such that

- unprimed entries $T_1$ weakly increase across each row from left to right
- strictly increase down each column from top to bottom

and primed entries

- $T'_1$ strictly increase across each row from left to right
- weakly increase down each column from top to bottom

Ex: $m = 4, \ n = 6, \ \lambda = (5, 4, 2)$,
Supersymmetric Schur function

Since

\[ s_\lambda(x/y) = \sum_{\mu} s_\mu(x) s_{\lambda'/\mu'}(y) \]

with

\[ s_\mu(x) = \sum_{T \in T^\mu(m)} x^{\text{wgt}(T)} \]

and

\[ s_{\lambda'/\mu'}(y) = \sum_{T \in T^{\lambda'/\mu'}(n)} y^{\text{wgt}(T)} \]

we have

\[ s_\lambda(x/y) = \sum_{T \in T^{\lambda}(m/n)} (x y)^{\text{wgt}(T)} \]
Supersymmetric Schur function

- Since 
  \[ s_\lambda(x/y) = \sum_{\mu} s_\mu(x) s_{\lambda'/\mu'}(y) \]

  with 
  \[ s_\mu(x) = \sum_{T \in T^\mu(m)} x^{\text{wgt}(T)} \]

  and 
  \[ s_{\lambda'/\mu'}(y) = \sum_{T \in T^{\lambda'/\mu'}(n)} y^{\text{wgt}(T)} \]

  we have 
  \[ s_\lambda(x/y) = \sum_{T \in T^\lambda(m/n)} (x y)^{\text{wgt}(T)} \]

- Ex: \( m = 4, n = 6, \lambda = (5, 4, 2) \)

  \[
  \begin{array}{cccccc}
  1 & 3 & 3 & 3' & 4' \\
  4 & 1' & 2' & 3' \\
  1' & 5'
  \end{array}
  \]

  \[
  (xy)^{\text{wgt}(T)} = x_1 x_3^2 x_4 y_1'^2 y_2' y_3'^2 y_4' y_5'
  \]
Littlewood-Richardson coefficients

- Let \( x = (x_1, \ldots, x_m) \) with \( m \in \mathbb{N} \)

- In \( \Lambda_m \) the ring of symmetric polynomial functions

\[
s_{\lambda}(x) s_{\mu}(x) = \sum_{\nu} c_{\lambda \mu}^{\nu} s_{\nu}(x)
\]

where the coefficients \( c_{\lambda \mu}^{\nu} \) are non-negative integers – the Littlewood-Richardson coefficients
Littlewood-Richardson coefficients

Let \( x = (x_1, \ldots, x_m) \) with \( m \in \mathbb{N} \)

In \( \Lambda_m \) the ring of symmetric polynomial functions
\[
s_\lambda(x) s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x)
\]
where the coefficients \( c_{\lambda\mu}^\nu \) are non-negative integers – the Littlewood-Richardson coefficients

Let \( x = (x_1, \ldots, x_m), \ y = (y_1, \ldots, y_n) \) with \( m, n \in \mathbb{N} \)

In \( \Lambda_{(m/n)} \) the ring of supersymmetric polynomial functions
\[
s_\lambda(x/y) s_\mu(x/y) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x/y)
\]
where the same Littlewood-Richardson coefficients occur.
Constraints on supersymmetric Schur functions

Notice that

\[ s_\nu(x) = s_\nu(x_1, \ldots, x_m) = 0 \quad \text{if} \quad \lambda'_1 > m \quad \text{while} \]

\[ s_\nu(x/y) = s_\nu(x_1, \ldots, x_m/y_1, \ldots, y_n) = 0 \quad \text{if} \quad \lambda'_{n+1} > m \]
Constraints on supersymmetric Schur functions

- Notice that \( s_\nu(x) = s_\nu(x_1, \ldots, x_m) = 0 \) if \( \lambda'_1 > m \) while \( s_\nu(x/y) = s_\nu(x_1, \ldots, x_m/y_1, \ldots, y_n) = 0 \) if \( \lambda'_{n+1} > m \)

- that is \( s_\nu(x) \neq 0 \) iff \( F^\nu \) lies within a horizontal strip of depth \( m \)

\[ \text{Ex: } m = 4, \quad F^\nu = \]

- and \( s_\nu(x/y) \neq 0 \) iff \( F^\nu \) lies within a hook with arm width \( m \) and leg width \( n \)

\[ \text{Ex: } m = 2, \quad n = 3, \quad F^\nu = \]
Supersymmetric row and column restricted identities

- With respect to the bases $s_\lambda(x)$ and $s_\lambda(x/y)$ the rings $\Lambda_n$ and $\Lambda_{(m/n)}$ coincide modulo the horizontal strip and hook shape restrictions on $\lambda$

- It follows that any identity expressed in terms of Schur functions $s_\lambda(x)$ takes exactly the same form in terms of supersymmetric Schur functions $s_\lambda(x/y)$

- However, the generating functions for Schur function series require amendment for the corresponding supersymmetric Schur function series
Supersymmetric Schur function series

\[ \sum_{\lambda} s_{\lambda}(x/y) = \frac{\prod_i \prod_a (1 + x_i y_a)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k) \prod_a (1 - y_a) \prod_{b<c} (1 - y_b y_c)} \]

\[ \sum_{\lambda: \ell(\lambda') \leq p} s_{\lambda}(x/y) \]

\[ = \frac{\prod_i \prod_a (1 + x_i y_a) \sum_{\mu \in \mathcal{P}_p} (-1)^{[\mu] - r(\mu)(p-1)/2} s_{\mu}(x/y)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k) \prod_a (1 - y_a) \prod_{b<c} (1 - y_b y_c)} \]

\[ \sum_{\lambda: \ell(\lambda) \leq p} s_{\lambda}(x/y) \]

\[ = \frac{\prod_i \prod_a (1 + x_i y_a) \sum_{\mu \in \mathcal{P}_{-p}} (-1)^{[\mu] - r(\mu)(p-1)/2} s_{\mu}(x/y)}{\prod_i (1 - x_i) \prod_{j<k} (1 - x_j x_k) \prod_a (1 - y_a) \prod_{b<c} (1 - y_b y_c)} \]
Supersymmetric Schur function series

\[
\sum_{\lambda \text{ even}} s_{\lambda}(x/y) = \frac{\prod_i \prod_{a} (1 + x_i y_a)}{\prod_{j \leq k} (1 - x_j x_k) \prod_{b < c} (1 - y_b y_c)}
\]

\[
= \sum_{\lambda \text{ even} : \ell(\lambda') \leq 2p} s_{\lambda}(x/y) \sum_{\mu \in \mathcal{P}_{2p+1}} (-1)^{|\mu| - r(\mu)(2p)/2} s_{\mu}(x/y)
\]

\[
= \prod_i \prod_{a} (1 + x_i y_a) \frac{\sum_{\mu \in \mathcal{P}_{-2p-1}} (-1)^{|\mu| - r(\mu)(2p)/2} s_{\mu}(x/y)}{\prod_{j \leq k} (1 - x_j x_k) \prod_{b < c} (1 - y_b y_c)}
\]

\[
= \sum_{\lambda' \text{ even} : \ell(\lambda) \leq 2p} s_{\lambda}(x/y)
\]
Supersymmetric Schur function series

\[ \sum_{\lambda' \text{ even}} s_{\lambda}(x/y) = \frac{\prod_i \prod_a (1 + x_i y_a)}{\prod_{j < k} (1 - x_j x_k) \prod_{b \leq c} (1 - y_b y_c)} \]

\[ \sum_{\lambda' \text{ even} : \ell(\lambda') \leq p} s_{\lambda}(x/y) = \frac{\prod_i \prod_a (1 + x_i y_a) \sum_{\mu \in \mathcal{P}_{p-1} : r(\mu) \text{ even}} (-1)^{|\mu| - r(\mu)p}/2 s_{\mu}(x/y)}{\prod_{j < k} (1 - x_j x_k) \prod_{b \leq c} (1 - y_b y_c)} \]

\[ \sum_{\lambda \text{ even} : \ell(\lambda) \leq p} s_{\lambda}(x/y) = \frac{\prod_i \prod_a (1 + x_i y_a) \sum_{\mu \in \mathcal{P}_{-p+1} : r(\mu) \text{ even}} (-1)^{|\mu| - r(\mu)p}/2 s_{\mu}(x/y)}{\prod_{j \leq k} (1 - x_j x_k) \prod_{b < c} (1 - y_b y_c)} \]
Supersymmetric form of the Cauchy identities

Let \( x = (x_1, \ldots, x_m), \ y = (y_1, \ldots, y_n), \ z = (z_1, \ldots, z_d), \ w = (w_1, \ldots, w_e) \) with \( m, n, d, e \in \mathbb{N} \), then

\[
\sum_{\lambda} s_{\lambda}(x/y) \ s_{\lambda}(z/w) = \frac{\prod_{j,l} (1 + x_i w_l) \ \prod_{j,k} (1 + y_j z_k)}{\prod_{i,k} (1 - x_i z_k) \ \prod_{j,l} (1 - y_j w_l)}
\]

\[
\sum_{\lambda: \ell(\lambda') \leq p} s_{\lambda}(x/y) \ s_{\lambda}(z/w) = \sum_{\lambda} s_{\lambda}(x/y) \ s_{\lambda}(z/w) \cdot \sum_{\zeta} (-1)^{|\zeta|} s_{\zeta+p^r}(x/y) \ s_{\zeta'+p^r}(z/w)
\]

\[
\sum_{\lambda: \ell(\lambda) \leq p} s_{\lambda}(x/y) \ s_{\lambda}(z/w) = \sum_{\lambda} s_{\lambda}(x/y) \ s_{\lambda}(z/w) \cdot \sum_{\eta} (-1)^{|\eta|} s_{\eta-p^r}(x/y) \ s_{\eta'+p^r}(z/w)
\]
Dual pairs of Lie supergroups

- Howe’s original work on dual pairs encompassed Lie supergroups, such as $GL(m/n)$ and $OSp(m/n)$

- Thus all our supersymmetric identities should be placed within this context

- They may be derived from the following dual pairs [Cheng and Zhang 04, Kwon 08]
Dual pairs of Lie supergroups

Howe’s original work on dual pairs encompassed Lie supergroups, such as $GL(m/n)$ and $OSp(m/n)$.

Thus all our supersymmetric identities should be placed within this context.

They may be derived from the following dual pairs [Cheng and Zhang 04, Kwon 08]

**Ex:** Dual pairs supercentralising one another in the given module

\[
\begin{align*}
S(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d/e}) & : GL(m/n) \times GL(d/e) \\
\Lambda(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d/e}) & : GL(m/n) \times GL(d/e) \\
S(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d}) & : OSp(m/n) \times O(d) \\
S(\mathbb{C}^{m/n} \otimes \mathbb{C}^{d}) & : OSp(m/n) \times Sp(d)
\end{align*}
\]
Jacobi-Trudi identities

- So far we have discussed infinite series of Schur functions, including their expression in determinantal form.
- These have been used to provide generalisations of formulae of both Littlewood and Cauchy.
- All the formulae have arisen from the expression of a Schur function as ratio of two alternants.
- It is natural to ask if similar results can be obtained from the expression of a Schur function in Jacobi-Trudi form.
For partitions $\lambda$ and $\mu$, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i$.

If $\mu \subseteq \lambda$ then the skew Young diagram $F^{\lambda/\mu}$ is defined to be $F^\lambda \backslash F^\mu$.

Ex: $F^{5443/431}$

\[ \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array} \]
Jacobi-Trudi identities

- For partitions $\lambda$ and $\mu$, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i$.
- If $\mu \subseteq \lambda$ then the skew Young diagram $F_{\lambda/\mu}^\lambda$ is defined to be $F_{\lambda}^\lambda \setminus F_{\mu}^\mu$.

Ex: $F_{5443/431}^{5443} = \left\{ \begin{array}{c}
\end{array} \right.$

Schur functions:

$$ s_\lambda(x) = \left| h_{\lambda_i - i + j}(x) \right| = \left| s_{\lambda_i - i + j}(x) \right|, $$
Jacobi-Trudi identities

- For partitions $\lambda$ and $\mu$, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all $i$.

- If $\mu \subseteq \lambda$ then the skew Young diagram $F^{\lambda/\mu}$ is defined to be $F^\lambda \setminus F^\mu$.

Ex: $F^{5443/431} =$

Schur functions:

$$s_\lambda(x) = \left| h_{\lambda_i - i + j}(x) \right| = \left| s_{\lambda_i - i + j}(x) \right|,$$

Skew Schur functions:

$$s_{\lambda/\mu}(x) = \left| h_{\lambda_i - \mu_j - i + j}(x) \right| = \left| s_{\lambda_i - \mu_j - i + j}(x) \right|,$$
Bressoud and Wei [1992] For all integers $t \geq -1$:

$$2^{(t-\lfloor t \rfloor)/2} \left| h_{\lambda_i-i+j}(x) + (-1)^{(t+\lfloor t \rfloor)/2} h_{\lambda_i-i-j+1-t}(x) \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[\sigma]+r(\sigma)(\lfloor t \rfloor-1)/2} s_{\lambda/\sigma}(x)$$
Bressoud-Wei identities

- **Bressoud and Wei [1992]** For all integers $t \geq -1$:

$$2^{(t-|t|)/2} \left| h_{\lambda_i-i+j}(x) + (-1)^{(t+|t|)/2} h_{\lambda_i-i-j+1-t}(x) \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[\sigma]+r(\sigma)(|t|-1)/2} s_{\lambda/\sigma}(x)$$

- **Hamel and K [2008]** For all integers $t$ and all $q$:

$$\left| h_{\lambda_i-i+j}(x) + q \chi_{j>-t} h_{\lambda_i-i-j+1-t}(x) \right|$$

$$= \sum_{\sigma \in \mathcal{P}_t} (-1)^{[\sigma]-r(\sigma)(t+1)/2} q^{r(\sigma)} s_{\lambda/\sigma}(x)$$
Algebraic proof

\[
\left| h_{\lambda_i-i+j}(x) + q \chi_j > t h_{\lambda_i-i-j+1-t}(x) \right| \\
= \sum_{r=0}^{n} \sum_{\kappa} q^r \left| h_{\lambda_i-i+j-\kappa_j}(x) \right| \\
= \sum_{\sigma \in \mathcal{P}_i} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)} q^r \left| h_{\lambda_i-i+j-\sigma_j}(x) \right|
\]
Algebraic proof

\[ |h_{\lambda_i-i+j}(x) + q \chi_{j>t} h_{\lambda_i-i-j+1-t}(x)| \]

\[ = \sum_{r=0}^{\infty} \sum_{\kappa} q^r |h_{\lambda_i-i+j-\kappa_j}(x)| \]

\[ = \sum_{\sigma \in \mathcal{P}_t} (-1)^{(j_r-1)+\cdots+(j_2-1)+(j_1-1)} q^r |h_{\lambda_i-i+j-\sigma_j}(x)| \]

- \( \kappa_j = 2j-1+t \) for \( j \in \{j_1, j_2, \ldots, j_r\} \) and \( \kappa_j = 0 \) otherwise
- with \( n \geq j_1 > j_2 > \cdots > j_r \geq 1 - \chi_{t<0} \)
- \( \sigma = \begin{pmatrix} j_1 - 1 + t & j_2 - 1 + t & \cdots & j_r - 1 + t \\ j_1 - 1 & j_2 - 1 & \cdots & j_r - 1 \end{pmatrix} \in \mathcal{P}_t \)
- \( r = r(\sigma) \)
Combinatorial proof

Lattice path interpretation of determinant

\[
\begin{align*}
&\left| h_{\lambda_i - i + j}(x) + q \chi_j > -t h_{\lambda_i - i - j + 1 - t}(x) \right| \\
= & \sum_{\pi \in S_n} (-1)^\pi \prod_{i=1}^n \left( h_{\lambda_i - i + \pi(i)}(x) + q \chi_{\pi(i)} > -t h_{\lambda_i - i - \pi(i) + 1 - t}(x) \right)
\end{align*}
\]
Combinatorial proof

- Lattice path interpretation of determinant

\[
\left| h_{\lambda_i-i+j}(x) + q \chi_{j=t} h_{\lambda_i-i-j+1-t}(x) \right|
\]

\[
= \sum_{\pi \in S_n} (-1)^\pi \prod_{i=1}^n \left( h_{\lambda_i-i+\pi(i)}(x) + q \chi_{\pi(i)>t} h_{\lambda_i-i-\pi(i)+1-t}(x) \right)
\]

- Each \( \pi \) defines a set of \( n \)-tuples of north-east paths
- For \( i = 1, 2, \ldots, n \) the \( i \)th path goes
  - from \( P_{\pi(i)} = (n + 1 - \pi(i), 1) \) or \( P'_{\pi(i)} = (n + t + \pi(i), 1) \)
  - to \( Q_i = (n + 1 + \lambda_i - i, n) \)
  - each step east at height \( k \) carries weight \( x_k \)
  - each path from \( P'_{\pi(i)} \) (rather than \( P_{\pi(i)} \)) carries weight \( q \)
An $n$-tuple of lattice paths

$\Pi = (1, 2, 3, 4)
\Lambda = (6, 4, 4, 2)$

Contribution

$(-1)^{t+0} q^2 (x_2) (1) (x_1 x_3^2) (x_3 x_4)$
An \( n \)-tuple of lattice paths

\[ \begin{align*}
  n &= 4, \quad t = 2 \\
  \lambda &= (6, 4, 4, 2) \\
  \pi &= \begin{pmatrix}
      1 & 2 & 3 & 4 \\
      3' & 1' & 2 & 4
    \end{pmatrix}
\end{align*} \]

\( \text{Ex.1} \)

Contribution

\[ (-1)^{2+0} q^2 (x_2) (1) (x_1 x_3^2) (x_3 x_4) \]
Combinatorial proof contd.

\[ \sum_{\pi \in S_n} (-1)^\pi \prod_{i=1}^{n} \left( h_{\lambda_i - i + \pi(i)(x)} + q \chi_{\pi(i) > -t} h_{\lambda_i - i - \pi(i) + 1 - t}(x) \right) \]
Combinatorial proof contd.

\[
\sum_{\pi \in S_n} (-1)^\pi \prod_{i=1}^n \left( h_{\lambda_i - i + \pi(i)}(x) + q \chi_{\pi(i) > t} h_{\lambda_i - i - \pi(i) + 1 - t}(x) \right)
\]

- Path from \( P_{\pi(i)} \) to \( Q_i \) contributes to \( h_{\lambda_i - i + \pi(i)}(x) \)
- Path from \( P'_{\pi(i)} \) to \( Q_i \) contributes to \( h_{\lambda_i - i - \pi(i) + 1 - t}(x) \)
- Sign changing involution removes contributions from intersecting paths
- All paths in \( n \)-tuple non-intersecting implies \( \pi = \)

\[
\begin{pmatrix}
1 & 2 & \cdots & r & r + 1 & r + 2 & \cdots & n \\
\pi(1) & \pi(2) & \cdots & \pi(r) & \pi(r + 1) & \pi(r + 2) & \cdots & \pi(n)
\end{pmatrix}
\]

with \( \pi(1) > \pi(2) > \cdots > \pi(r) \) for \( P'_{\pi(i)} Q_i \) paths
and \( \pi(r + 1) < \pi(r + 2) < \cdots < \pi(n) \) for \( P_{\pi(i)} Q_i \) paths
Combinatorial proof contd.

1. Eastward distance $P_i$ to $Q_i = \lambda_i$ for $i = 1, \ldots, n$
2. Let distance $P_i$ to $P'_{\pi(i)} = \sigma_i$ for $i = 1, \ldots, r$
3. Let distance $P_i$ to $P_{\pi(i)} = \sigma_i$ for $i = r + 1, \ldots, n$
4. Then, in Frobenius notation

$$\sigma = \begin{pmatrix}
\pi(1) - 1 + t & \pi(2) - 2 + t & \cdots & \pi(r) - r + t \\
\pi(1) - 1 & \pi(2) - 2 & \cdots & \pi(r) - r \\
\end{pmatrix}$$
Combinatorial proof contd.

- Eastward distance $P_i$ to $Q_i = \lambda_i$ for $i = 1, \ldots, n$
- Let distance $P_i$ to $P'_{\pi(i)} = \sigma_i$ for $i = 1, \ldots, r$
- Let distance $P_i$ to $P_{\pi(i)} = \sigma_i$ for $i = r + 1, \ldots, n$
- Then, in Frobenius notation
  \[
  \sigma = \begin{pmatrix}
  \pi(1) - 1 + t & \pi(2) - 2 + t & \cdots & \pi(r) - r + t \\
  \pi(1) - 1 & \pi(2) - 2 & \cdots & \pi(r) - r \\
  \end{pmatrix}
  \]
- Now re-interpret $i$th path monomial as contribution to $i$th row of an $s_{\lambda/\sigma}(x)$ semistandard tableau, so that our determinant reduces to
  \[
  \sum_{\sigma \in \mathcal{P}_t} (-1)^{(\pi(r)-1)+\cdots+(\pi(2)-1)+(\pi(1)-1)} q^r \left| h_{\lambda_i - i + j - \sigma_j(x)} \right|
  \]
  as required
Semistandard skew tableaux

Each $n$-tuple of non-intersecting paths defines a semistandard skew tableaux

**Ex. 1**

$n = 4$, $t = 2$,

$\lambda = (6, 4, 4, 2)$

$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3' & 1' & 2 & 4 \end{pmatrix}$
Semistandard skew tableaux

Each $n$-tuple of non-intersecting paths defines a semistandard skew tableaux

Ex.1

$n = 4, \ t = 2,$

$\lambda = (6, 4, 4, 2)$

$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3' & 1' & 2 & 4 \end{pmatrix}$

$\mu = (5, 4, 1) = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix} \in \mathcal{P}_2$
An $n$-tuple of lattice paths

Ex.2

$n = 4$, $t = -2$

$\lambda = (5, 4, 4, 3, 3, 2)$

$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5' & 3' & 1 & 2 & 4 & 6 \end{pmatrix}$

Contribution

$(-1)^{4+2} q^2 (x_1x_6) (x_1x_2) (x_3^2) (x_4) (x_3x_6) (x_1x_5)$
An $n$-tuple of lattice paths

Ex. 2

$n = 4, \quad t = -2$

$\lambda = (5, 4, 4, 3, 3, 2)$

$\pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
5' & 3' & 1 & 2 & 4 & 6
\end{pmatrix}$

Contribution

$(-1)^{4+2} q^2 (x_1 x_6) (x_1 x_2) (x_3^2) (x_4) (x_3 x_6) (x_1 x_5)$
Semistandard skew tableaux

Ex. 2

\[ n = 4, \quad t = -2, \]
\[ \lambda = (5, 4, 4, 3, 3, 2) \]
\[ \pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
5' & 3' & 1 & 2 & 4 & 6
\end{pmatrix} \]
Semistandard skew tableaux

Ex. 2

\[ n = 4, \quad t = -2, \]
\[ \lambda = (5, 4, 4, 3, 3, 2) \]
\[ \pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
5' & 3' & 1 & 2 & 4 & 6
\end{pmatrix} \]

\[
\begin{array}{cccccccccc}
& & & & & & & & & & \\
& 5 & & & & & & & & \\
& & 4 & & & & & & & \\
& 3 & & 3 & & 3 & & 3 & & \\
& 1 & & 1 & & 1 & & 1 & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 1 & 6 \\
0 & 0 & 1 & 2 \\
0 & 0 & 3 & 3 \\
0 & 0 & 4 \\
0 & 3 & 6 \\
1 & 5 \\
\end{array}
\]

\[ \mu = (3, 2, 2, 2, 1) = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix} \in \mathcal{P}_{-2} \]
Cauchy-type Jacobi-Trudi expansion

Theorem [Hamel and K, 2008]

- Let \( x = (x_1, \ldots, x_m) \) and \( y = (y_1, \ldots, y_n) \)
- Let \( \lambda \) and \( \mu \) have lengths \( \ell(\lambda) \leq m \) and \( \ell(\mu) \leq n \)
- Then for each pair of integers \( p \) and \( q \) we have

\[
\left| \begin{array}{c}
h_{\mu_{n+1-i} + i-j}(y) \\
\vdots \\
\chi_{j \leq n+p} h_{\lambda_{i-n-i} + j-p}(x) \\
\end{array} \right| \quad : \quad \begin{array}{c}
\chi_{j > n-q} h_{\mu_{n+1-i} + i-j-q}(y) \\
\vdots \\
h_{\lambda_{i-n-i} + j}(x) \\
\end{array}
\]

\[
= \sum_{\zeta \subseteq n^m} (-1)^{||\zeta||} s_{\lambda/(\zeta + p r(\zeta))}(x) s_{\mu/(\zeta' + q r(\zeta))}(y)
\]
Cauchy-type extension

where the determinant is \((m + n) \times (m + n)\), and is partitioned after the \(n\)th row and \(n\)th column

and if \(\zeta = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix} \in (n^m)\)

with \(a_1 < n, \ b_1 < m\) and \(r = r(\zeta)\), then

\[
\zeta + p^r = \begin{pmatrix} a_1 + p & a_2 + p & \cdots & a_r + p \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}
\]

\[
\zeta' + q^r = \begin{pmatrix} b_1 + q & b_2 + q & \cdots & b_r + q \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}
\]

with \(a_r \geq \max\{0, -p\}\) and \(b_r \geq \max\{0, -q\}\)
Example

- $m = 3$, $n = 4$, $p = -2$, $q = -1$, $\lambda = (5, 3, 2)$, $\mu = (4, 3, 2, 2)$
- Let $\{k\} = h_k(x)$ and $\{k\} = h_k(y)$ for all integers $k$
Example

- $m = 3$, $n = 4$, $p = -2$, $q = -1$, $\lambda = (5, 3, 2)$, $\mu = (4, 3, 2, 2)$

- Let $\{k\} = h_k(x)$ and $\{k\} = h_k(y)$ for all integers $k$
Typical term in Laplace expansion

\[
\pi = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 3 & 4 & 6 & 2 & 5 & 7
\end{pmatrix}
\]

\((-1)^\pi = (-1)^{0+1+1+2} = +1\)
Typical term in Laplace expansion

\[ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 2 & 5 & 7 \end{pmatrix} \]

\[ (-1)^\pi = (-1)^{0+1+1+2} = +1 \]

\[
\begin{vmatrix}
\{2\} & \{0\} & - & - \\
\{3\} & \{1\} & \{0\} & - \\
\{5\} & \{3\} & \{2\} & \{1\} \\
\{7\} & \{5\} & \{4\} & \{3\}
\end{vmatrix}
\times
\begin{vmatrix}
\{4\} & \{5\} & \{7\} \\
\{1\} & \{2\} & \{4\} \\
- & \{0\} & \{2\}
\end{vmatrix}
\]
Typical term in Laplace expansion

\[ \pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 & 2 & 5 & 7 \end{pmatrix} \]

\[ (-1)^\pi = (-1)^{0+1+1+2} = +1 \]

\[ \begin{vmatrix} \{2\} & \{0\} & - & - \\ \{3\} & \{1\} & \{0\} & - \\ \{5\} & \{3\} & \{2\} & \{1\} \\ \{7\} & \{5\} & \{4\} & \{3\} \end{vmatrix} \times \begin{vmatrix} \{4\} & \{5\} & \{7\} \\ \{1\} & \{2\} & \{4\} \\ - & \{0\} & \{2\} \end{vmatrix} \]

\[ \left( \zeta_1', \ldots, \zeta_1' \mid \zeta_1, \ldots, \zeta_m \right) = \left( 1 - 1, 3 - 2, 4 - 3, 6 - 4 \mid 5 - 2, 6 - 5, 7 - 7 \right) = (0, 1, 1, 2 \mid 3, 1, 0) \]

\[ \zeta = (3, 1, 0) = \begin{pmatrix} 5 - 2 & -1 \\ 6 - 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = (-1)^{|\zeta|} = +1 \]

\[ r(\zeta) = 1 \]
Determination of $\zeta + p^r$ and $\zeta' + q^r$

\[ \zeta = (310) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad r = r(\zeta) = 1 \quad p = -2 \]

\[ \implies \zeta + p^r = (310) - (200) = (110) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \zeta' = (2110) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad r = r(\zeta) = 1 \quad q = -1 \]

\[ \implies \zeta' + q^r = (2110) - (1000) = (1110) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \]
Identification of constituent determinants

- Recall that $\lambda = (532)$ and $\mu = (4322)$
- while $\zeta + p^r = (110)$ and $\zeta' + q^r = (1110)$

$$\begin{array}{ccc}
\{4\} & \{5\} & \{7\} \\
\{1\} & \{2\} & \{4\} \\
- \{0\} & \{2\}
\end{array} = s_{532/110}(x)$$
Identification of constituent determinants

Recall that \( \lambda = (532) \) and \( \mu = (4322) \)

while \( \zeta + p^r = (110) \) and \( \zeta' + q^r = (1110) \)

\[
\begin{vmatrix}
\{4\} & \{5\} & \{7\} \\
\{1\} & \{2\} & \{4\} \\
- & \{0\} & \{2\}
\end{vmatrix} = s_{532/110}(x)
\]

\[
\begin{vmatrix}
\{2\} & \{0\} & - & - \\
\{3\} & \{1\} & \{0\} & - \\
\{5\} & \{3\} & \{2\} & \{1\} \\
\{7\} & \{5\} & \{4\} & \{3\}
\end{vmatrix} = s_{4322/1110}(y)
\]
Conclusions

Both the classical Schur function series of Littlewood and the Cauchy identity may be restricted with respect to row lengths or column lengths through determinantal formulae.

In each case the correction factors to the original multiplicative formulae may be expressed as a signed sum of Schur functions or pairs of Schur functions specified by partitions having a particularly simple form in Frobenius notation.

Each row or column restricted Schur function series is nothing other than the character of some (rather simple) finite or infinite-dimensional irrep of a classical group.
Conclusions

- To evaluate these characters (and thereby derive the restricted Schur function series) use may be made of Howe dual pairs with respect to spin and metaplectic representations of (the covering groups) of the orthogonal and symplectic groups.

- All the Schur function identities may be extended to the case of supersymmetric Schur functions.

- The dual pair approach enables many other characters to be evaluated, although in doing so it is usually necessary to invoke classical group modification rules.
Some references on background and motivation


Row length restricted series


Row length restricted series


Group characters and modification rules


Complementary groups or dual pairs


Dual pairs in spin modules


Dual pairs in metaplectic modules


The analogy between spin and metaplectic modules

Dual pairs in supersymmetric modules


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