

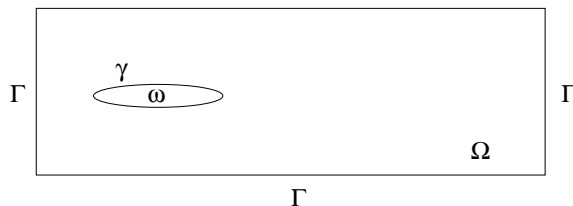
# On a Domain Embedding Method for Flow around Moving Rigid Bodies

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## 1 Introduction

Several applications lead to the numerical simulation of incompressible viscous flow around moving rigid bodies; let us mention for example blood flow around artificial heart valves. In this article we consider only the case where the rigid body motions are known a priori; the more complicated case where the rigid body motions are caused by hydrodynamical forces, among other forces, will be discussed in a forthcoming article. Following an approach advocated — to our knowledge — by Peskin [Pes72] we use a *domain embedding* method (also called *fictitious domain method* by some authors) which consists of filling the moving bodies by the surrounding fluid and taking into account the boundary conditions on these bodies by introducing a well chosen distribution of boundary forces. In the particular case of the *Dirichlet boundary* conditions considered in this article it is quite convenient to use a *Lagrange multiplier* method which is well suited to the variational formulations commonly used to study the Navier-Stokes equations and their approximation, by finite element methods for example. Another important component of the solution method is a time discretization by operator splitting which reduces the simulation to a sequence of subproblems for which efficient solution methods exist already.

Figure 1. The flow region  
 $\Gamma$



## 2 A Model Problem and its Lagrange Multiplier/Domain Embedding Formulation

The geometrical situation is as in Figure 1. With  $\omega = \omega(t)$  a moving rigid body ( $\omega \subset \Omega \subset \mathbf{R}^d$ ,  $d = 2, 3$ ), we consider for  $t > 0$  the solution of the *Navier-Stokes equations*

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega \setminus \overline{\omega(t)}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \setminus \overline{\omega(t)}, \quad (2)$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus \overline{\omega(0)}, \text{ (with } \nabla \cdot \mathbf{u}_0 = 0), \quad (3)$$

$$\mathbf{u} = \mathbf{g}_0 \text{ on } \Gamma, \quad (4)$$

$$\mathbf{u} = \mathbf{g}_1 \text{ on } \gamma(t). \quad (5)$$

In (1)-(5)  $\mathbf{u}$  and  $p$  denote, as usual, the *velocity* and *pressure*, respectively;  $\nu (> 0)$  is the *viscosity*,  $\mathbf{f}$  the density of external forces,  $\mathbf{x}$  the generic point of  $\mathbf{R}^d$  ( $\mathbf{x} = \{x_i\}_{i=1}^d$ ),  $\gamma(t) = \partial\omega(t)$  and  $(\mathbf{u} \cdot \nabla) \mathbf{u} = \{\sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}\}_{i=1}^d$ . We suppose that  $\mathbf{g}_1$  is the velocity on  $\gamma(t)$  of the rigid body  $\omega(t)$  which implies that  $\int_{\gamma(t)} \mathbf{g}_1 \cdot \mathbf{n} d\gamma = 0$ , and that  $\int_{\Gamma} \mathbf{g}_0 \cdot \mathbf{n} d\Gamma = 0$ . In the following, we shall use, if necessary, the notation  $\phi(t)$  for the function  $\mathbf{x} \rightarrow \phi(\mathbf{x}, t)$ .

We introduce first the functional spaces  $\mathbf{V}_{\mathbf{g}_0(t)} = \{\mathbf{v} | \mathbf{v} \in (H^1(\Omega))^d, \mathbf{v} = \mathbf{g}_0(t) \text{ on } \Gamma\}$ ,  $\mathbf{V}_0 = (H_0^1(\Omega))^d$ ,  $L_0^2(\Omega) = \{q | q \in L^2(\Omega); \int_{\Omega} q d\mathbf{x} = 0\}$  and  $\Lambda(t) = (\mathbf{H}^{-1/2}(\gamma(t)))^d$ . With  $\tilde{\mathbf{f}}$  an  $L^2$ -lifting of  $\mathbf{f}$  in  $\Omega$  (we can take  $\tilde{\mathbf{f}}|_{\overline{\omega(t)}} = \mathbf{0}$ ) and  $\nabla \cdot \mathbf{U}_0 = 0$  ( $\mathbf{U}_0|_{\Omega \setminus \overline{\omega(0)}} = \mathbf{u}_0$ ), it can be shown — at least formally — that problem (1)-(5) is *equivalent* to

$$\begin{aligned} \text{For } t \geq 0, \text{ find } \{\mathbf{U}(t), P(t), \lambda(t)\} \in \mathbf{V}_{\mathbf{g}_0(t)} \times L_0^2(\Omega) \times \Lambda(t) \text{ such that} \\ \int_{\Omega} \frac{\partial \mathbf{U}}{\partial t} \cdot \mathbf{v} d\mathbf{x} + \nu \int_{\Omega} \nabla \mathbf{U} \cdot \nabla \mathbf{v} d\mathbf{x} + \int_{\Omega} (\mathbf{U} \cdot \nabla) \mathbf{U} \cdot \mathbf{v} d\mathbf{x} - \int_{\Omega} P \nabla \cdot \mathbf{v} d\mathbf{x} \\ = \int_{\Omega} \tilde{\mathbf{f}} \cdot \mathbf{v} d\mathbf{x} + \int_{\gamma(t)} \lambda \cdot \mathbf{v} d\gamma, \quad \forall \mathbf{v} \in \mathbf{V}_0, \end{aligned} \quad (6)$$

$$\int_{\Omega} q \nabla \cdot \mathbf{U}(t) d\mathbf{x} = 0, \quad \forall q \in L^2(\Omega), \quad (7)$$

$$\int_{\gamma(t)} (\mathbf{U}(t) - \mathbf{g}_1(t)) \cdot \mu d\gamma = 0, \quad \forall \mu \in \Lambda(t), \quad (8)$$

$$\mathbf{U}(0) = \mathbf{U}_0 \text{ in } \Omega, \quad \mathbf{U} = \mathbf{g}_0 \text{ on } \Gamma, \quad (9)$$

in the sense that  $\mathbf{U}(t)|_{\Omega \setminus \overline{\omega(t)}} = \mathbf{u}(t)$  and  $P(t)|_{\Omega \setminus \overline{\omega(t)}} = p(t)$ . We can easily show that  $\lambda = [\nu \partial \mathbf{U} / \partial \mathbf{n} - \mathbf{n} P]_{\gamma}$ , where  $[ \ ]_{\gamma}$  denotes the *jump* at  $\gamma$ .

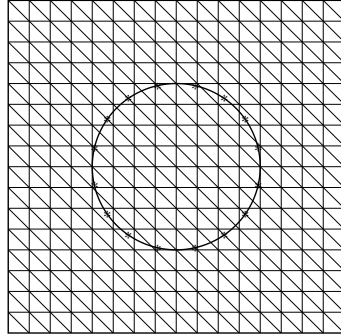
*Remark 2.1:* The mathematical analysis of flow problems such as (1)-(5) is addressed in, e.g., [AG93] (see also the references therein).

*Remark 2.2:* We observe that the *actual geometry*, i.e.,  $\omega(t)$  and  $\gamma(t)$  occurs “only” in the  $\gamma(t)$ -integral in (6) and in (8); this is a justification of the domain embedding

approach.

### 3 Finite Element Approximation of Problem (6)-(9)

Figure 2. Part of the triangulation of  $\Omega$  with mesh points indicated by “\*” on the disk boundary



We suppose that  $\Omega \subset \mathbf{R}^2$  ( $d = 2$ ). With  $h$  a space discretization step we introduce a *finite element* triangulation  $\mathcal{T}_h$  of  $\bar{\Omega}$  and then  $\mathcal{T}_{h/2}$  a triangulation twice finer obtained by joining the midpoints of the edges of  $\mathcal{T}_h$ . We define then the following finite dimensional spaces which approximate  $\mathbf{V}_{\mathbf{g}_0}$ ,  $\mathbf{V}_0$ ,  $L^2(\Omega)$ ,  $L_0^2(\Omega)$  respectively

$$\mathbf{V}_{\mathbf{g}_{0h}} = \{\mathbf{v}_h | \mathbf{v}_h \in C^0(\bar{\Omega})^2, \mathbf{v}_h|_T \in P_1 \times P_1, \forall T \in \mathcal{T}_h, \mathbf{v}_h|_\Gamma = \mathbf{g}_{0h}\}, \quad (10)$$

$$\mathbf{V}_{0h} = \{\mathbf{v}_h | \mathbf{v}_h \in C^0(\bar{\Omega})^2, \mathbf{v}_h|_T \in P_1 \times P_1, \forall T \in \mathcal{T}_h, \mathbf{v}_h|_\Gamma = \mathbf{0}\}, \quad (11)$$

$$L_h^2 = \{q_h | q_h \in C^0(\bar{\Omega}), q_h|_T \in P_1, \forall T \in \mathcal{T}_{2h}\}, L_{0h}^2 = \{q_h | q_h \in L_h^2, \int_\Omega q_h \, dx = 0\}; \quad (12)$$

in (10)-(12),  $P_1$  is the space of the polynomials in  $x_1, x_2$  of degree  $\leq 1$  and  $\mathbf{g}_{0h}$  is an approximation of  $\mathbf{g}_0$  such that  $\int_\Gamma \mathbf{g}_{0h} \cdot \mathbf{n} \, d\Gamma = 0$ . Concerning the space  $\Lambda_h(t)$  approximating  $\Lambda(t)$ , we define it by

$$\Lambda_h(t) = \{\mu_h | \mu_h \in (L^\infty(\gamma(t)))^2, \mu_h \text{ is constant on the arc joining } 2 \text{ consecutive mesh points on } \gamma(t)\}. \quad (13)$$

A particular choice for the mesh points on  $\gamma$  is visualized on Figure 2, where  $\omega$  is a disk. Let us resist any requirement that the mesh points on  $\gamma$  have to be at the intersection of  $\gamma$  with the triangle edges of  $\mathcal{T}_{h/2}$ ; (see [GG95] for more details and the relations between  $h_\Omega$  and  $h_\gamma$ ). This kind of decoupling between the  $\Omega$  and  $\gamma$  meshes makes the domain embedding approach attractive for problems with moving boundaries like those discussed in this note. With the above spaces it is natural to approximate problem (6)-(9) by (with obvious notation)

$$\begin{aligned} & \int_\Omega \frac{\partial \mathbf{U}_h}{\partial t} \cdot \mathbf{v} \, dx + \nu \int_\Omega \nabla \mathbf{U}_h \cdot \nabla \mathbf{v} \, dx + \int_\Omega (\mathbf{U}_h \cdot \nabla) \mathbf{U}_h \cdot \mathbf{v} \, dx - \int_\Omega P_h \nabla \cdot \mathbf{v} \, dx \\ & = \int_\Omega \tilde{\mathbf{f}}_h \cdot \mathbf{v} \, dx + \int_{\gamma(t)} \lambda_h \cdot \mathbf{v} \, d\gamma, \forall \mathbf{v} \in \mathbf{V}_{0h}, \mathbf{U}_h(t) \in \mathbf{V}_{\mathbf{g}_0(t)h}, \end{aligned} \quad (14)$$

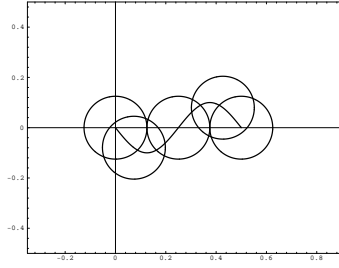
$$\int_{\Omega} q \nabla \cdot \mathbf{U}_h(t) \, d\mathbf{x} = 0, \forall q \in L_h^2, P_h(t) \in L_{0h}^2, \tag{15}$$

$$\int_{\gamma(t)} (\mathbf{U}_h(t) - \mathbf{g}_1(t)) \cdot \boldsymbol{\mu} \, d\gamma = 0, \forall \boldsymbol{\mu} \in \Lambda_h(t), \lambda_h(t) \in \Lambda_h(t), \tag{16}$$

$$\mathbf{U}_h(0) = \mathbf{U}_{0h}; \tag{17}$$

in (17),  $\mathbf{U}_{0h}$  is an approximation of  $\mathbf{U}_0$ , approximately divergence-free.

Figure 3.



#### 4 Time Discretization of (14)-(17) by Operator Splitting Methods

From an abstract point of view problem (14)-(17) is a particular case of the following class of initial value problems

$$\frac{d\phi}{dt} + A_1(\phi) + A_2(\phi) + A_3(\phi) = f, \phi(0) = \phi_0, \tag{18}$$

where the operators  $A_i$  can be *multivalued*. Among many operator splittings which can be employed to solve (18) we advocate the very simple one below (analyzed in, e.g., [Mar90]); it is only first-order accurate but its low order accuracy is compensated by good stability and robustness properties.

*A fractional step scheme à la Marchuk-Yanenko:* With  $\Delta t$  a time discretization step and the initial guess,  $\phi^0 = \phi_0$ , the scheme is defined as follows:

For  $n \geq 0$ , we obtain  $\phi^{n+1}$  from  $\phi^n$  via the solution of

$$(\phi^{n+j/3} - \phi^{n+(j-1)/3})/\Delta t + A_j(\phi^{n+j/3}) = f_j^{n+1}, \tag{19}$$

with  $j = 1, 2, 3$  and  $\sum_{j=1}^3 f_j^{n+1} = f^{n+1}$ . Applying scheme (19) to problem (14)-(17) we obtain (with  $0 \leq \alpha, \beta \leq 1, \alpha + \beta = 1$ , and after dropping some of the subscripts  $h$ ):

$$\mathbf{U}^0 = \mathbf{U}_{0h}; \tag{20}$$

for  $n \geq 0$ , we compute  $\{\mathbf{U}^{n+1/3}, P^{n+1/3}\}$ ,  $\mathbf{U}^{n+2/3}$ ,  $\{\mathbf{U}^{n+1}, \lambda^{n+1}\}$  via the solution of

$$\begin{cases} \int_{\Omega} \frac{\mathbf{U}^{n+1/3} - \mathbf{U}^n}{\Delta t} \cdot \mathbf{v} \, d\mathbf{x} - \int_{\Omega} P^{n+1/3} \nabla \cdot \mathbf{v} \, d\mathbf{x} = 0, \forall \mathbf{v} \in \mathbf{V}_{0h}, \\ \int_{\Omega} q \nabla \cdot \mathbf{U}^{n+1/3} \, d\mathbf{x} = 0, \forall q \in L_h^2; \mathbf{U}^{n+1/3} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1}, P^{n+1/3} \in L_{0h}^2, \end{cases} \quad (21)$$

$$\begin{cases} \int_{\Omega} \frac{\mathbf{U}^{n+2/3} - \mathbf{U}^{n+1/3}}{\Delta t} \cdot \mathbf{v} \, d\mathbf{x} + \alpha \nu \int_{\Omega} \nabla \mathbf{U}^{n+2/3} \cdot \nabla \mathbf{v} \, d\mathbf{x} \\ + \int_{\Omega} (\mathbf{U}^{n+1/3} \cdot \nabla) \mathbf{U}^{n+2/3} \cdot \mathbf{v} \, d\mathbf{x} = \alpha \int_{\Omega} \tilde{\mathbf{f}}^{n+1} \cdot \mathbf{v} \, d\mathbf{x}, \forall \mathbf{v} \in \mathbf{V}_{0h}; \\ \mathbf{U}^{n+2/3} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1}, \end{cases} \quad (22)$$

$$\begin{cases} \int_{\Omega} \frac{\mathbf{U}^{n+1} - \mathbf{U}^{n+2/3}}{\Delta t} \cdot \mathbf{v} \, d\mathbf{x} + \beta \nu \int_{\Omega} \nabla \mathbf{U}^{n+1} \cdot \nabla \mathbf{v} \, d\mathbf{x} \\ = \beta \int_{\Omega} \tilde{\mathbf{f}}^{n+1} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\gamma^{n+1}} \lambda^{n+1} \cdot \mathbf{v} \, d\gamma, \forall \mathbf{v} \in \mathbf{V}_{0h}, \\ \int_{\gamma^{n+1}} (\mathbf{U}^{n+1} - \mathbf{g}_{1h}^{n+1}) \cdot \boldsymbol{\mu} \, d\gamma = 0, \forall \boldsymbol{\mu} \in \Lambda_h^{n+1}; \\ \mathbf{U}^{n+1} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1} (= \mathbf{V}_{\mathbf{g}_0((n+1)\Delta t)h}), \lambda^{n+1} \in \Lambda_h^{n+1} (= \Lambda_h((n+1)\Delta t)). \end{cases} \quad (23)$$

## 5 Solution of the Subproblems (21), (22) and (23)

By inspection of (21) it is clear that  $\mathbf{U}^{n+1/3}$  is the  $L^2(\Omega)^2$ -projection of  $\mathbf{U}^n$  on the (affine) subset of the functions  $\mathbf{v} \in \mathbf{V}_{\mathbf{g}_{0h}}^{n+1}$  such that  $\int_{\Omega} q \nabla \cdot \mathbf{v} \, d\mathbf{x} = 0$ ,  $\forall q \in L_h^2$ ,  $P^{n+1/3}$  being the corresponding Lagrange multiplier in  $L_{0h}^2$ . The pair  $\{\mathbf{U}^{n+1/3}, P^{n+1/3}\}$  is *unique* and to compute it we can use an Uzawa/conjugate gradient algorithm operating in  $L_{0h}^2$  equipped with the scalar product  $\{q, q'\} \rightarrow \int_{\Omega} \nabla q \cdot \nabla q' \, d\mathbf{x}$ . We obtain thus an algorithm preconditioned by the discrete equivalent of  $-\Delta$  for the homogeneous Neumann boundary condition. Such an algorithm is *very* easy to implement and is described in [GPP96]; it seems to have excellent convergence properties.

If  $\alpha > 0$ , problem (22) is a classical one; it can be easily solved, for example, by a least squares/conjugate gradient algorithm like those discussed in [Glo84].

If  $\beta > 0$  the solution of problem (23) has been discussed in [GPP94]. In the particular case where  $\beta = 0$ , problem (23) reduces to an  $L^2(\Omega)^2$ -projection over the subspace of  $\mathbf{V}_{\mathbf{g}_{0h}}^{n+1}$  of the functions  $\mathbf{v}$  satisfying the condition  $\int_{\gamma^{n+1}} (\mathbf{v} - \mathbf{g}_{1h}^{n+1}) \cdot \boldsymbol{\mu} \, d\gamma = 0$ ,  $\forall \boldsymbol{\mu} \in \Lambda_h^{n+1}$ . It follows from the above observation that if  $\beta = 0$ , problem (23) can be solved by an Uzawa/conjugate gradient algorithm operating in  $\Lambda_h^{n+1}$ , which has many similarities with the algorithm used to solve problem (21). If one uses the trapezoidal rule to compute the various  $L^2(\Omega)$ -integrals in (23), taking  $\beta = 0$  brings further simplification since in that particular case  $\mathbf{U}^{n+1}$  will coincide with  $\mathbf{U}^{n+2/3}$  at those vertices of  $\mathcal{T}_{h/2}$  such that the support of the related shape function does not intersect  $\gamma^{n+1}$ ; from the above observation it follows that to obtain  $\mathbf{U}^{n+1}$  and  $\lambda^{n+1}$  we have to solve a linear system of the following form

$$\mathbf{Ax} + \mathbf{B}^t \mathbf{y} = \mathbf{b}, \quad \mathbf{Bx} = \mathbf{c}. \quad (24)$$

For the numerical simulations presented in Section 6 we have used  $\alpha = 1$  and  $\beta = 0$  in (22), (23).

## 6 Numerical Experiments

We simulate a two-dimensional flow with  $\Omega = (-0.35, 0.9) \times (-0.5, 0.5)$  (see Figure 3) and  $\omega$  a moving disk of radius 0.125. The center of the disk is moving between  $(0, 0)$  and  $(0.5, 0)$  along a prescribed trajectory  $(x(t), y(t)) = (0.25(1 - \cos(\frac{\pi t}{2})), -0.1 \sin(\pi(1 - \cos(\frac{\pi t}{2}))))$  (see Figure 3) of period 4. Several different positions of the disk have been shown on Figure 3. The boundary conditions are  $\mathbf{u} = \mathbf{0}$  on  $\Gamma$  and  $\mathbf{u}$  on  $\partial\omega(t)$  coinciding with the disk velocity. We suppose that the disk rotates counterclockwise at angular velocity  $2\pi$ . Since we are taking  $\nu = 0.005$ , the maximum Reynolds number based on the disk diameter as characteristic length is 102.336. On  $\Omega$  we have used a regular triangulation  $\mathcal{T}_{h/2}$  to approximate the velocity, like the one in Figure 2, the pressure grid  $\mathcal{T}_h$  being twice coarser. Concerning  $\Lambda_h(t)$ ,  $\gamma(t)$  has been divided into  $M$  subarcs of equal length. We have done two simulations: For the first one we have taken  $h = 1/128$ ,  $\Delta t = 0.00125$  and  $M = 80$ . For the second we have taken  $h = 1/256$ ,  $\Delta t = 0.00125$  and  $M = 160$ . With stopping criteria of the order of  $10^{-12}$  we need around 10 iterations at most to have convergence of the conjugate gradient algorithms used to solve the problems at each step of the scheme (20)-(23). On Figure 4, we show the isobar lines, the vorticity density and the streamlines obtained at  $t = 5, 6, 7, 8$  for  $h = 1/256$ ,  $\Delta t = 0.00125$  and  $M = 160$ . There is a good agreement between the results obtained from these two simulations.

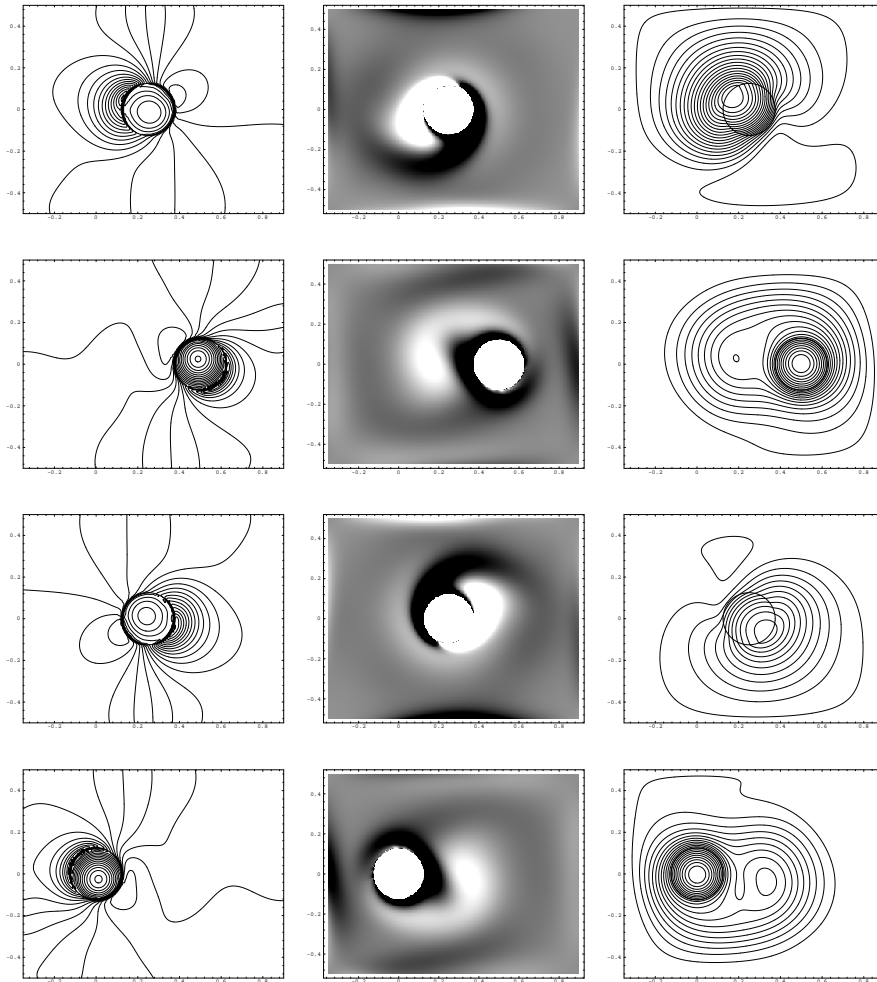
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Figure 4. Isobar lines (at left), vorticity density (at middle) and streamlines (at right) at time  $t = 5, 6, 7, 8$  in one period of disk motion. The disk moves from the left to the right, then to the left. The mesh size for velocity (resp., pressure) is  $h = 1/256$  (resp.,  $h = 1/128$ ).



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